Mathematical Reviews

Edited by

O. Neugebauer **M. H. Stone** **O. Veblen**

R. P. Boas, Jr., ***Executive Editor***

Vol. 7, No. 8 **September, 1946** **pp. 353–404**

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MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
l'Intermédiaire des Recherches Mathématiques
Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima
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Editorial Office

Mathematical Reviews, Brown University, Providence 12, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to Mathematical Reviews, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 7, No. 8 SEPTEMBER, 1946 Pages 353-404

HISTORY

Bruins, E. M. On the approximation to $\pi/4$ in Egyptian geometry. Nederl. Akad. Wetensch., Proc. 48, 206–210 = Indagationes Math. 7, 11–15 (1945). (Dutch) [MF 15791]

Neugebauer, O. The history of ancient astronomy: Problems and methods. Publ. Astr. Soc. Pacific 58, 17–43, 104–142 (1946). [MF 16299]

Slightly enlarged republication of a survey which appeared in Journal of Near Eastern Studies 4, 1–38 (1945); cf. these Rev. 6, 141.

Katô, Heizaemon. Investigations of Seki-Kôwa's Kaihô-Hompen. Tôhoku Math. J. 48, 1–24 (1941). (Japanese) [MF 16342]

The work in question is concerned with the theory of equations.

Minoda, Takashi. On "Katuyô Sampô, Book III" of T. Seki. II. Tôhoku Math. J. 48, 167–173 (1941). (Japanese) [MF 16356]

Minoda, Takashi. On "Keimen Endan" of M. Araki. Tôhoku Math. J. 48, 174–184 (1941). (Japanese) [MF 16357]

Part I of the first paper appeared in the same J. 47, 99–109 (1940); these Rev. 1, 289. The works in question deal with trigonometry.

Fujiwara, Matsusaburô. Miscellaneous notes on the history of Chinese mathematics. IV. (Mathematics in the old Korea. II.) Tôhoku Math. J. 48, 78–88 (1941). (Japanese) [MF 16352]

The preceding part appeared in the same J. 47, 309–321 (1940); these Rev. 2, 306.

Fujiwara, Matsusaburô. Miscellaneous notes on the history of Wazan. X. The works of Yosizane Tanaka. Tôhoku Math. J. 49, 90–105 (1942). (Japanese) [MF 14699]

★Cherniss, Harold. The Riddle of the Early Academy. University of California Press, Berkeley and Los Angeles, 1945. vi+103 pp. $1.50.

The prominent role which mathematical concepts played in Plato's philosophical system has frequently been discussed. Special interest arose from the story, reported by Aristoxenus, that Plato's lecture "On the good" disappointed the audience because of its largely mathematical content. The relationship between ideas and numbers was more and more considered to be a cornerstone of Plato's real theory, not promulgated in his public writings. Far-reaching consequences were drawn from this for the character of Greek mathematics in the Academy. This book is a critical analysis of this situation. Its result is the denial of any secret oral tradition of a Platonic philosophy and the statement that the dialogues contain all that Plato considered essential concerning the theory of ideas.

O. Neugebauer (Princeton, N. J.).

★Rome, A. Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste. Tome III. Théon d'Alexandrie. Commentaire sur les Livres 3 et 4 de l'Almageste. Studi e Testi 106. Biblioteca Apostolica Vaticana, Roma, 1943. xxvi+279 pp. [paged cxv–cxl+807–1085].

This is a continuation of the Greek text of the commentaries on the Almagest by Pappus, Theon and Hypatia. Volumes I and II contained commentaries on books 1 and 2 [same series 72, 1936] and on books 5 and 6 [same series 54, 1931]. The present volume thus completes the commentaries concerning the theory of the moon. In addition to the general introduction, a large apparatus of explanatory notes is given below the text. *O. Neugebauer.*

Bell, Eric Temple. Sixes and sevens. Scripta Math. 11, 139–171 (1945). [MF 14661]

Continued from page 50 of the same volume. See these Rev. 6, 253 for the preceding three parts. *O. Neugebauer.*

Junge, Gustav. Die pythagoreische Zahlenlehre. Deutsche Math. 5, 341–357 (1940). [MF 14340]

The author deals with the various aspects which the integers play in Greek philosophy. All the facts are well known.

O. Neugebauer (Princeton, N. J.).

★Wavre, Rolin. Les apories de Zénon d'Elée. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 123–127, Füssli, Zürich, 1945.

Frajese, Attilio. Su un passo geometrico controverso del "Menone." Boll. Un. Mat. Ital. (2) 5, 182–189 (1943). [MF 16105]

Frajese, Attilio. Talete di Mileto e le origini della geometria greca. Boll. Un. Mat. Ital. (2) 4, 49–60 (1942). [MF 16056]

★de la Harpe, Jean. Les progrès de l'idée du temps dans la philosophie grecque. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 128–137, Füssli, Zürich, 1945.

Casara, Giuseppina. Un problema archimedeo di terzo grado e le sue soluzioni attraverso i tempi. Boll. Un. Mat. Ital. (2) 4, 244–262 (1942). [MF 16076]

Equazione relativa al problema di Archimede "dividere una sfera data in due parti aventi fra loro un dato rapporto."

Esame delle soluzioni dei Greci, degli Arabi, di Huygens, della scuola napoletana, e considerazioni generali.
Author's summary.

Ḥijāb, Muḥammad 'Ali, and Sidrāk, Ṣubḥi. Study on al-Khāzini and his book "The Balance of Wisdom." Proc. Math.-Phys. Soc. Egypt **2**, no. 1, Arabic pp. 1–15 (1941). (Arabic)

Excerpts from the work quoted in the title. Cf. H. Suter, Die Mathematiker und Astronomen der Araber und ihre Werke, Abh. Gesch. Math. Wiss. **10** (1900), p. 226.

Chakaia, D. G. Trigonometry of the peoples of the Near East in one of the Georgian monuments of astronomical literature. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **13**, 207–219 (1944). (Georgian. Russian summary) [MF 14625]

According to the Russian summary, the author studies questions of Arabic trigonometry in a Georgian translation of a work of Ulug-Beg.

Bortolotti, Ettore. La pubblicazione delle opere e del carteggio matematico di Paolo Ruffini. Boll. Un. Mat. Ital. (2) **5**, 114–120 (1943). [MF 16096]

Carruccio, Ettore. Galileo precursore della teoria degli insiemi. Boll. Un. Mat. Ital. (2) **4**, 175–187 (1942). [MF 16069]

van Haaften, M. Quelques nouvelles données concernant l'histoire des anciennes tables néerlandaises de logarithmes. Nieuw Arch. Wiskunde (2) **21**, 59–64 (1941). [MF 15713]

★Scholz, Heinrich. Pascals Forderungen an die mathematische Methode. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 19–33, Füssli, Zürich, 1945.

Fleckenstein, J. O. Die Taylorsche Formel bei Johann I Bernoulli. Elemente der Math. **1**, 13–17 (1946). [MF 15692]

★Spiess, Otto. Die Summe der reziproken Quadratzahlen. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 66–86, Füssli, Zürich, 1945.

A historical article discussing various methods of evaluating the series in question. Particular attention is given to a very simple proof by Niklaus I Bernoulli, hitherto unpublished.

Spiess, O. Über einige neu aufgefundene Schriften der alten Basler Mathematiker. Verh. Naturforsch. Ges. Basel **56**, 86–111 (1945). [MF 15413]

In dem Manuskript Mss. Inv. 1607 der Genfer Universitätsbibliothek entdeckte der Verfasser eine Anzahl von nachgelassenen teilweise nicht veröffentlichten Schriften der Basler Mathematiker Jakob, Johann, Niklaus I und Daniel Bernoulli und Jakob Hermann. Unbekannt waren bisher die hier angetroffenen Fragmente von Vorlesungen Jakob Bernoulli's über Experimentalphysik aus den Jahren 1683–1690 sowie 12 akademische Reden von Hermann. Das Manuskript stammt aus der Bibliothek des Genfer Physikers Jean Jallabert, der die Stücke während eines Studienaufenthaltes in Basel im Winter 1737 kopieren liess.
E. J. Dijksterhuis (Oisterwijk).

★Carathéodory, C. Basel und der Beginn der Variationsrechnung. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 1–18, Füssli, Zürich, 1945.

In this brief account of the important contributions to the early history of the calculus of variations which are due to mathematicians born in Basel, the author makes some very incisive remarks. The following is an example: "James Bernoulli had the ability to create values in mathematics which, like coined gold, remain at par during all times." While largely concerned with John and James Bernoulli and Euler, the article also says a good deal about Lagrange; it throws an interesting light, moreover, on Huygens's significance for the development of the subject.
A. Dresden (Swarthmore, Pa.).

★Ackeret, J. Leonhard Eulers letzte Arbeit. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 160–168, Füssli, Zürich, 1945.

Hofmann, Jos. E. Weiterbildung der logarithmischen Reihe Mercators in England. III. Halley, Moivre, Cotes. Deutsche Math. **5**, 358–375 (1940). [MF 14342]

[For parts I and II cf. Deutsche Math. **3**, 598–605 (1938); **4**, 556–562 (1939); these Rev. **1**, 33.] This is the final instalment of an account of the development of the theory and use of logarithms in England from 1668 to about 1740. Part III covers the life and work of Halley, G. Anderson, de Moivre and Cotes, concluding with a short summary of the whole investigation and a complete index. The central idea is the use of the power series for log $(1+x)$, which was first published by Mercator (1668). Significant steps taken by de Moivre and Cotes are well brought out. It would have been appropriate to include Maclaurin also. The summary makes no mention of work, by J. Gregory and others, involving the integration of log x.
H. W. Turnbull.

van Veen, S. C. Historical details concerning $\int_0^\infty e^{-x^2}dx$. Mathematica, Zutphen. B. **12**, 1–4 (1943). (Dutch) [MF 14314]

Whittaker, E. T. The sequence of ideas in the discovery of quaternions. Revista Unión Mat. Argentina **11**, 4–10 (1945). (Spanish) [MF 14097]

Translation of an article in Proc. Roy. Irish Acad. Sect. A. **50**, 93–98 (1945); these Rev. **6**, 254.

Riabouchinsky, Dimitri. Le rôle de la mécanique des fluides dans le développement de la théorie des fonctions d'une variable complexe. C. R. Acad. Sci. Paris **222**, 426–428 (1946). [MF 16019]

★Beth, E. W. Geschiedenis der Logica. [History of Logic]. N. V. Servire, 's Gravenhage, 1944. 96 pp. (Dutch)

Im Altertum behandelt der Verfasser Antisthenes, Euklid von Megara, Plato und (einigermassen ausführlich im Anschluss an seine eigenen Untersuchungen) Aristoteles und seine Schüler und, recht ausführlich, Chrysippos. Das Mittelalter wird ziemlich kurz besprochen. In der Neuzeit behandelt der Verfasser Jungius, Descartes, Hobbes, Ath. Kircher, John Wilkins, Leibniz (recht ausführlich), Lambert, Kant; von den modernsten, Boole, Jevons, Peirce, Schröder, Peano, Frege, Russell, Hilbert, Brouwer und ganz kurz das Allerneuste. Im Anhang Untersuchungen von I. M. Bochensky über antike und mittelalterliche Logik.
H. Freudenthal (Amsterdam).

Beth, E. W. Logistic as extension of the traditional forma logic. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 34, 53–68 (Annalen van het Genootschap voor Wetenschappelijke Philosophie 11, 1–16) (1940). (Dutch) [MF 15811]

Der Verfasser gibt eine Übersicht über die Entwicklung der formalen Logik von Aristoteles bis zur Logistik. Sehr interessant sind verschiedene historische Fussnoten.

H. Freudenthal (Amsterdam).

Kuypers, K. Two types of progress with respect to the logic of Aristotle. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 37, 31–40 (1943). (Dutch) [MF 15809]

Mit den beiden Fortschritten hinsichtlich der Aristotelischen Logik meint der Verfasser einerseits die moderne logistische Einstellung, andererseits die moderne Aristotelesforschung, deren Ergebnisse (Stenzel und Solmsen) der Verfasser skizziert. *H. Freudenthal* (Amsterdam).

de Losada y Puga, Cristóbal. Copernicus. Revista Univ. Católica Perú 11, 149–178 (1943). (Spanish) [MF 14186]

The three chapters of this article are entitled Astronomy at the time of Copernicus, The life and work of Copernicus, The Copernican system and modern science.

de Losada y Puga, Cristóbal. The third centenary of Newton. Revista Univ. Católica Perú 10, 479–480 (1942). (Spanish) [MF 14187]

★Bernštein, S. N., Editor. Naučnoe Nasledie P. L. Čebyševa. Vypusk Pervyi: Matematika. [The Scientific Legacy of P. L. Čebyšev. First Part: Mathematics]. Academiya Nauk SSSR, Moscow-Leningrad, 1945. 174 pp. (1 plate) (Russian)

The book contains essays on various aspects of Čebyšev's mathematical work and the later development of the topics discussed. The topics are: Orthogonal polynomials, by N. I. Ahiezer; Probability, by S. N. Bernštein; Theory of numbers, by I. M. Vinogradov and B. N. Delone; Integration of algebraic functions, by V. V. Golubev; Best approximation, by V. L. Gončarov.

★Aleksandrov, P. S., and Kolmogorov, A. N. Nikolaĭ Ivanovič Lobačevskiĭ, 1793–1943. OGIZ, Moscow-Leningrad, 1943. 100 pp. (Russian)

The book consists of three parts, the first being biographical and dealing briefly with Lobačevskiĭ's scientific spirit and his influence on the science and culture of the day. The second part is a modern exposition of his geometry; the axioms of Archimedes and of Cantor are introduced and freedom from contradiction from Euclidean and Lobačevskian geometry are established. The last part deals with the influence of Lobačevskiĭ's work on the thought of the nineteenth century. It served as a point of departure from "absolute" geometry to the geometrical work of Riemann and thus had a profound influence on the physical concepts of the universe that culminated in the theory of relativity.

M. S. Knebelman (Pullman, Wash.).

Wilson, Edwin B. Obituary: George David Birkhoff. Science (N.S.) 102, 578–580 (1945).

Obituary: George David Birkhoff (1884–1944). Revista Soc. Cubana Ci. Fís. Mat. 2, 28–29 (1945). (Spanish) [MF 14183]

Rey Pastor, J. Professor George D. Birkhoff and his influence in Argentina. Gaz. Mat., Lisboa 6, no. 26, 12–13 (1945). (Spanish) [MF 15117]

Reprinted from Revista Ci., Lima 47, 105–109 (1945); these Rev. 6, 254.

Labra, Manuel. Biographical note: Jean Baptiste Joseph Fourier (1768–1830). Revista Soc. Cubana Ci. Fís. Mat. 2, 24–26 (1945). (Spanish) [MF 14182]

Milne, E. A. Obituary: Ralph Howard Fowler. J. London Math. Soc. 19, 244–256 (1944). [MF 14575]

Milne, E. A. Obituary: Ralph Howard Fowler, 1889–1944. Obit. Notices Roy. Soc. London 5, 61–78 (1 plate) (1945). [MF 15680]

The list of mathematical papers by Prof. M. Fujiwara. Tôhoku Math. J. 49, 133–138 (1942). [MF 14704]

van Veen, S. C. In memoriam David Hilbert (1862–1943). Mathematica, Zutphen. B. 11, 159–169 (1943). (Dutch) [MF 14313]

Bouligand, Georges. Obituary: Émile Picard. Rev. Gén. Sci. Pures Appl. 52, 1–3 (1942). [MF 14546]

Lefort, Guy. Obituary: Émile Picard. Revista Mat. Hisp.-Amer. (4) 5, 147–151 (1945). (Spanish) [MF 15204]

Semple, J. G. Obituary: Charles Henry Rowe. J. London Math. Soc. 19, 241–244 (1944). [MF 14574]

Carslaw, H. S., and Hardy, G. H. Obituary: John Raymond Wilton. J. London Math. Soc. 20, 58–64 (1945). [MF 15935]

FOUNDATIONS

Weyl, Hermann. Mathematics and logic. A brief survey serving as preface to a review of "The Philosophy of Bertrand Russell." Amer. Math. Monthly 53, 2–13 (1946). [MF 15305]

Bouligand, G. Les crises de l'unité dans la mathématique. Rev. Gén. Sci. Pures Appl. 52, 215–221 (1945). [MF 16141]

Blumberg, Henry. On the change of form. Amer. Math. Monthly 53, 181–192 (1946). [MF 16144]

Kondô, Motokiti. Une méthode opérationnelle dans la théorie des nombres naturels. Proc. Imp. Acad. Tokyo 20, 564–568 (1944). [MF 14926]

This paper develops the theory of addition and multiplication of natural numbers upon the basis of the Peano postulates. The method used is not essentially new. Addition is defined as a certain linear operation, a linear operation being any operation F such that $F(x')=(F(x))'$ for all x, where x' is the successor of x. A similar development of multiplication is given in terms of additive operations, that is, operations F such that $F(x+y)=F(x)+F(y)$ for all x

and y. The proofs of the fundamental laws of addition and multiplication then emerge quite simply. [Erratum: p. 565, l. 13, for '$\phi_1(x)$' read '$\phi_y(x)$.'] *R. M. Martin.*

de Beauregard, Olivier Costa. Extension d'une théorie de M. J. de Neumann au cas des projecteurs non commutables. C. R. Acad. Sci. Paris 221, 230–231 (1945). [MF 14256]

The author suggests that a logic with three truth values is needed in quantum mechanics. Quantum mechanical propositions requiring only two truth values are called categorical. They correspond to projection operators in Hilbert space with a two-element spectrum. The author studies the logical conjunction of two categorical propositions with noncommutative projection operators. This logical conjunction is not categorical, and may be either true, false, or doubtful. *O. Frink* (State College, Pa.).

Destouches-Février, Paulette. Sur les rapports entre la logique et la physique théorique. C. R. Acad. Sci. Paris 219, 481–483 (1944). [MF 15280]

The author elaborates on the thesis of G. Birkhoff and J. von Neumann [Ann. of Math. (2) 37, 823–843 (1936)] that certain physical theories, such as quantum mechanics, require a logic of nonclassical type. Propositions of quantum mechanics concerning the results of measurement correspond to closed linear subspaces of a linear topological space. Logical product of propositions corresponds to set intersection of subspaces, but logical sum does not correspond to set union, since the union of subspaces need not be a subspace. The classical Boolean logic can be used only in the case where all quantities are, in theory, simultaneously measurable. *O. Frink* (State College, Pa.).

Destouches-Février, Paulette. Logique adaptée aux théories quantiques. C. R. Acad. Sci. Paris 221, 287–288 (1945). [MF 14501]

This note describes very briefly a logic of quantum mechanics more general than that of G. Birkhoff and J. von Neumann [Ann. of Math. (2) 37, 823–843 (1936)]. This new logic has two addition operations, called strong logical sum and weak logical sum. With respect to weak logical sum, the system is a lattice which is complemented and modular, but not distributive. With respect to strong logical sum, the system is a distributive lattice, but is not complemented. The law of excluded middle holds for weak logical sum, but not for strong logical sum. The logical operations correspond to operations on closed linear subspaces of Hilbert space. *O. Frink* (State College, Pa.).

Bochvar, D. A. Some logical theorems on the normal sets and predicates. Rec. Math. [Mat. Sbornik] N.S. 16(58), 345–352 (1945). (Russian. English summary) [MF 14587]

In a previous paper [same Rec. N.S. 15(57), 369–384 (1945); these Rev. 7, 46] the author indicated a noncontradictory system K_0 of axioms for the functional calculus with quantified predicates. In K_0 the formula $(\varphi)(P(\varphi)\to\overline{\varphi(\varphi)})\to\overline{P(P)}$ is provable. Extensions of this formula, such as

$$(\varphi)[(E\psi_1)\cdots(E\psi_n)(P(\psi_1)\wedge\cdots\wedge\psi_n(\varphi))\to\overline{\varphi(\varphi)}]\to\overline{P(P)}$$

are discussed. Furthermore, an infinite sequence $\{\mathfrak{A}_n(Q)\}$ of formulas is given such that the adjunction to K_0 of $F(Q)=_D\mathfrak{A}_n(Q)$ as the definition of a predicate F renders the system contradictory. Here $\mathfrak{A}_0=\overline{Q(Q)}$; from $F(Q)=_D\mathfrak{A}_1=(\varphi)(Q(\varphi)\to\overline{\varphi(\varphi)})$, $F(F)$ as well as $\overline{F(F)}$ is explicitly derived; the interesting point is that from these formulas F can be eliminated by application of the definition of F, so that the criticism of Behmann [Jber. Deutsch. Math. Verein. 40, 37–48 (1931)] cannot be applied here. *A. Heyting* (Laren).

Beth, E. W. On formal logic and logic of content. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 37, 20–30 (1943). (Dutch) [MF 15808]

In einer Logik-Nummer anlässlich des 100-jährigen Jubiläum von Mills "System of Logic" betont der Verfasser den Gegensatz zwischen formaler und inhaltlicher Logik. Er übersieht als einen einheitlichen Entwicklungsgang die Geschichte der formalen Logik von Aristoteles über Kant bis zur modernen Logistik, welche drei Etappen er in ansprechender Weise skizziert. Zu jeder formalen Logik gehört nach dem Verfasser eine Inhaltslogik, die von dem Verfasser definiert wird als Interpretations-System der formalen Logik, etwa hinsichtlich der Anwendung der formalen Logik auf die Wirklichkeit. So geben die logischen Gesetze bei Aristoteles die Seins-Struktur der Wirklichkeit wieder, während sie bei Kant die notwendigen Bedingungen für das Zustandekommen sinnvoller Ergebnisse bilden. Ein Beispiel für Inhaltslogik am Rande der modernen Logistik sind für den Verfasser die Untersuchungen über die Relationen zwischen den drei Sprachsystemen der Quantenmechanik, die verschiedene Mathematiker und Physiker in den letzten Jahren konstatiert haben.

Die Auffassung und Definition der Inhaltslogik, die der Verfasser vertritt, wird bei aller verdienstlichen Schärfe, mit der er sie entwickelt, natürlich den Bestrebungen Mills in keiner Weise gerecht. Es handelt sich hier um inhaltliche Untersuchungen über die mathematische und physikalische Logik, aber kaum um eine wirkliche inhaltliche Logik, die gerade erst einmal die mathematische Einseitigkeit der formell logischen Methoden überwinden müsste. *H. Freudenthal* (Amsterdam).

Beth, E. W. Chapters from the modern formal logic. Euclides 18, 93–107 (1941); 19, 63–86, 147–160 (1942). (Dutch) [MF 15690]

The chapter headings are Logic of Propositions, Logic of Properties, Logic of Relations, Historical Remarks, Categories.

Vredenduin, P. G. J. Burkamp's logic. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 37, 41–51 (Annalen van het Genootschap voor Wetenschappelijke Philosophie 14, 1–11) (1943). (Dutch) [MF 15810]

In Burkamps Logik ist die Wahrheit dem Kriterium der "Richtigkeit" unterworfen; die Logik ist nicht autonom, sondern normativ; verschiedene logistische Systeme nebeneinander sind nicht möglich; die Funktion der Negation ist von der in der Logistik weitgehend verschieden. Der Verfasser kritisiert das System Burkamps. [Siehe W. Burkamp, "Wirklichkeit und Sinn," Junker und Dünnhaupt, Berlin, 1938.] *H. Freudenthal* (Amsterdam).

Bergmann, Gustav. Some comments on Carnap's logic of induction. Philos. Sci. 13, 71–78 (1946). [MF 15903]

The papers commented on appeared in Philos. Sci. 12, 72–97 (1945); Philos. and Phenomenonol. Res. 5, 513–532 (1945); these Rev. 7, 46, 189.

Hempel, Carl G. A note on the paradoxes of confirmation. Mind 55, 79–82 (1946). [MF 15333]

von Wright, Georg Henrik. Über Wahrscheinlichkeit. Eine logische und philosophische Untersuchung. Acta Soc. Sci. Fennicae. Nova Ser. A. 3, no. 11, 66 pp. (1945). [MF 14656]

The first of the three chapters of this work presents a system of axioms for the theory of probability and proves some well-known elementary theorems. Besides the usual symbols of logic and arithmetic, the author employs the expression "$P(A, H, p)$," which is to be read "the probability of the sentence A relative to the sentence H is p." In addition to the formal axioms involving this notion, there are also provided special rules of transformation; these rules, unfortunately, are so vaguely and loosely formulated as to be almost incomprehensible: the author himself refers to one of them as "etwas dunkle." The arguments of this chapter are only partially formalized; indeed, it is not easy to see how some of them (especially those in § 17) could be represented at all within the limits of the formal system which the author has described.

The second chapter is concerned with the interpretation and application of the formal system developed in the first. The author asserts that his system can be given a frequency interpretation; but that such is the case seems by no means obvious to the reviewer, especially in view of the fact that the author does not attempt to show that his transformation rules are satisfied by such an interpretation (and such a demonstration would probably present difficulties). It should be remarked that definition I (p. 33) is an incorrect definition of the notion it is intended to formulate (the notion: "p is the limit, as n approaches infinity, of the relative frequency with which A occurs in H_n"), because the given definition would imply that, for some N and q, we have $P(A, H_n, q)$ whenever $n > N$.

The final chapter is taken up with the question of defining the probability of hypotheses. By "hypothesis" the author means here simply a sentence which involves quantifiers. No important results seem to be obtained. This chapter, like the preceding one, is very difficult to understand because of the carelessness of its formulations.

J. C. C. McKinsey (Reno, Nev.).

van Dantzig, D. Mathematical and empirical foundations of the calculus of probability. Nederl. Tijdschr. Natuurkunde 8, 70–93 (1941). (Dutch. English summary)

By the axiomatic treatment of Reichenbach-Kolmogoroff (given here in a slightly different form) the mathematical problem of the foundation of the probability-calculus is completely separated from the corresponding empirical problem and can now be regarded as solved. Though some authors are of a contrary opinion, the paper tries to show that an axiomatic treatment of an empirical science does not solve the corresponding empiristic problem.

From the author's summary.

Churchman, C. W. Probability theory. I. Background. Philos. Sci. 12, 147–157 (1945). [MF 13929]

Churchman, C. W. Probability theory. II. Postulates of experimental method. Philos. Sci. 12, 158–164 (1945). [MF 13930]

Churchman, C. W. Probability theory. III. Non-mechanical concepts. Philos. Sci. 12, 165–173 (1945). [MF 13931]

Speiser, Andreas. Die räumliche Deutung der Aussenwelt. Actes Soc. Helv. Sci. Nat. 121, 38–51 (1941). [MF 15882]

ALGEBRA

Nandi, H. K. On the relation between certain types of tactical configurations. Bull. Calcutta Math. Soc. 37, 92–94 (1945). [MF 15684]

The author outlines a proof of the impossibility of the (15, 21, 7, 5, 2) incomplete balanced block configuration. His proof is based on a result by Q. M. Hussain stating that the (22, 22, 7, 7, 2) configuration is impossible. Hussain's result has, however, not yet been published in full.

H. B. Mann (Columbus, Ohio).

Hussain, Q. M. Symmetrical incomplete block designs with $\lambda=2$, $k=8$ or 9. Bull. Calcutta Math. Soc. 37, 115–123 (1945). [MF 15687]

The author considers incomplete block designs with $v=b=1+\frac{1}{2}k(k-1)$, $\lambda=2$. By a tactical enumeration which is outlined in the paper he found that no solution exists for $k=8$. For $k=9$ he finds two new solutions not isomorphic to the solution published in the statistical tables of Fisher and Yates.

H. B. Mann (Columbus, Ohio).

Schmidt, Karl. Stabilität und Aperiodizität bei Bewegungsvorgängen vierter Ordnung. Arch. Elektrotechnik 37, 217–220 (1943). [MF 15630]

This note concerns the solutions of the equation

$$A_4s^4+s^3+A_2s^2+A_1s+1=0,$$

where the A's are positive constants. For a fixed value of A_4, the relation between A_1 and A_2 is determined which makes two of the roots pure imaginary. Similarly, the relation between A_1 and A_2 is determined which makes two of the roots real, negative and equal. The results are presented graphically in eight diagrams.

L. A. MacColl.

Pipping, Nils. Verallgemeinerung der Cardanischen Formel. Acta Acad. Aboensis 13, no. 13, 9 pp. (1942). [MF 15100]

Skolem [Norsk Mat. Tidsskr. 12, 70–81 (1930)] has shown that the solution of the equation $x^5+5px^3+5p^2x+2q=0$ can be expressed in the form $x=y+z=u_1^{1/5}+u_2^{1/5}$, where $u_1, u_2=-q\pm(q^2+p^5)^{1/2}$ and $yz=-p$. The further extension of this generalisation of Cardan's formula to the case of equations of the form

$$x^{2k+1}+(2k+1)px^{2k-1}+a_1x^{2k-3}+a_2x^{2k-5}+\cdots+a_{k-1}x+2q=0$$

is studied. Explicit expressions for the coefficients a_i (in terms of p) are obtained which permit a representation of a root of the equation as the sum of two $(2k+1)$th roots. It is shown that in the cases when $k=(1, 2)$, 3 or 4 all the roots of the equation can be obtained by multiplying the two $(2k+1)$th roots by suitable $(2k+1)$th roots of unity. It is not known whether this result is true for larger values of k.

O. Todd-Taussky (Teddington).

Rados, Gustav. Die Faktorenzerlegung einiger komplizierter Polynome aus der Theorie der Kegelschnitte und Flächen zweiter Ordnung. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 749–764 (1940). (Hungarian. German summary) [MF 15557]

Rados, Gustav. Die Faktorenzerlegung zweier komplizierter n-variabliger Polynome. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 765–774 (1940). (Hungarian. German summary) [MF 15558]

Cherubino, Salvatore. Segnatura, divisori elementari e forme canoniche di una matrice. Boll. Un. Mat. Ital. (2) 4, 38–48 (1942). [MF 16055]

As a substitute for the Jordan canonical form of a matrix, the author advocates a form due to Predella and himself. He rediscovers the relationship between the Segre characteristic and the "signature of Scorza," which is actually the Weyr characteristic. *C. C. MacDuffee* (Madison, Wis.).

Bonferroni, C. E. Una disuguaglianza sui determinanti e il teorema di Hadamard. Boll. Un. Mat. Ital. (2) 4, 158–165 (1942). [MF 16066]

If A is a Hermitian matrix with first element a_{11} and the minor A_{n-1} of a_{11} is positive definite, then $\det A \leqq a_{11} \det A_{n-1}$. The equality holds when and only when a_{11} is the only nonzero element in the first row and column. In particular, the product of a complex matrix M and its conjugate transpose satisfies this condition. The theorem of Hadamard follows directly. The author deduces other inequalities and indicates an application in the theory of multiple correlation. *C. C. MacDuffee* (Madison, Wis.).

Everett, C. J., and Ryser, H. J. The Gram matrix and Hadamard theorem. Amer. Math. Monthly 53, 21–23 (1946). [MF 15308]

A vector space $V(C)$ over the complex field C is an inner product space if, for every ξ, η, ζ of V and a of C, there exists a complex number (ξ, η) such that $(\xi+\eta, \zeta)=(\xi, \zeta)+(\eta, \zeta)$, $(\xi, \eta a)=(\xi, \eta)a$, $(\xi, \eta)=\overline{(\eta, \xi)}$, $(\xi, \xi)\geqq 0$ and $(\xi, \xi)=0$ if and only if $\xi=0$. A set of vectors $(\delta_1, \cdots, \delta_n)$ is orthonormal if, for every i and j, (δ_i, δ_j) is 1 or 0 according as $i=j$ or $i\neq j$. If $(\xi_1, \cdots, \xi_n)$ are independent and span S, a set of n orthonormal vectors spanning S may be obtained by the Schmidt process, which involves a linear transformation of matrix T. The rank of the Gram matrix $G(\xi)=((\xi_i, \xi_j))$ is equal to the maximum number of independent vectors in the set $\xi_1, \cdots, \xi_n$. If this rank is n, $G(\xi)=T^*T$, $\det G(\xi)=|\det T|^2 \leqq (\xi_1, \xi_1)\cdots(\xi_n, \xi_n)$ (the Hadamard-Fischer theorem), equality holding if and only if the ξ_i are mutually orthogonal. It is shown incidentally that the Schwarz inequality is a special case of the Bessel inequality. *C. C. MacDuffee.*

Rados, Gustav. Die intuitive Herleitung einiger auf Hermitesche Determinanten bezüglichen Sätze. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 411–419 (1940). (Hungarian. German summary) [MF 15564]

Very simple proofs of well-known theorems concerning Hermitian determinants and their minors, using the corresponding Hermitian forms. *O. Szász* (Cincinnati, Ohio).

Duncan, W. J. Properties of characteristic numbers and modes deduced by matrix methods. Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. 2006 (8095 & 8447), 1–15 (1944). [MF 15948]

This is an expository article on the characteristic numbers and characteristic vectors of Hermitian, skew-Hermitian, orthogonal and unitary matrices. The author illustrates the theorems by means of simple examples and indicates their applications to dynamical problems. The statement on page 7 that the roots of $|H-\lambda C|=0$ are real "when H and C are Hermitian" should be amended to read "when H and C are Hermitian and one of them is definite." *J. Williamson* (Flushing, N. Y.).

Todd, Olga. A note on skew-symmetric matrices. Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. 2006 (8095 & 8447), 16–18 (1944). [MF 15949]

The author discusses the form of the matrices $X'X$ and XX', where X' is the transposed matrix of the matrix X whose columns are the characteristic vectors of a skew-symmetric matrix with distinct characteristic numbers. *J. Williamson* (Flushing, N. Y.).

Morita, Kiiti. Über normale antilineare transformationen. Proc. Imp. Acad. Tokyo 20, 715–720 (1944). [MF 14947]

An antilinear transformation $[A]$ is normal (commutative with its transpose) if and only if its matrix A is quasi-normal, that is, $A\bar{A}'=A'\bar{A}$. If U is unitary, A is said to be unitarily similar to $U'AU$. Then $U'AU$ is quasi-normal if and only if A is quasi-normal. Two quasi-normal matrices A and B are unitarily similar if and only if $A\bar{A}$ and $B\bar{B}$ are similar. A quasi-normal matrix A is unitarily similar to a direct sum of a diagonal matrix and blocks of the type

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

all elements in this canonical form being real and nonnegative. If a finite number of antilinear transformations are commutative with each other and with all their transposes, their matrices may be simultaneously reduced to such canonical forms. If $A\bar{A}$ is real, there is a real orthogonal matrix T such that $T'AT$ has the same canonical form as above except that the elements may be complex. An arbitrary complex matrix is unitarily similar to a triangular matrix with single elements or blocks of order two in the diagonal and 0's below these blocks. *C. C. MacDuffee.*

Pall, Gordon. Hermitian quadratic forms in a quasi-field. Bull. Amer. Math. Soc. 51, 889–893 (1945). [MF 14454]

The author proves the following generalization of a theorem of E. Witt [J. Reine Angew. Math. 176, 31–44 (1936)]. Let F be a quasi-field with a conjugate (an anti-automorphic involution), $2\neq 0$. Let a be a nonzero scalar and B_1, B_2 nonsingular Hermitian matrices of order $n-1$, with elements in F. Let

$$A_1=\begin{bmatrix} a & 0 \\ 0 & B_1 \end{bmatrix}, \quad A_2=\begin{bmatrix} a & 0 \\ 0 & B_2 \end{bmatrix}, \quad T=\begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix}$$

and $A_2=\bar{T}'A_1T$, where the elements of T are in F and T_1 is of order $n-1$. Then one can construct a transformation τ carrying B_1 into B_2. In fact, it is shown that one may take $\tau=T_1+xk\bar{x}'B_1T_1$, where k is a suitably chosen element of F. The author also proves that, if $\bar{z}'Az=a_{11}$, $\bar{A}'=A$, A nonsingular with elements in F, $a_{11}\neq 0$, then there exists in F a nonsingular automorph of A with z as its first column. The existence of τ is shown to be an immediate corollary of this result as well. *A. E. Ross* (St. Louis, Mo.).

Arf, Cahit. Untersuchungen über quadratische Formen in Körper der Charakteristik 2. II. Über aritmetische Aequivalenz quadratischer Formen in Potenzreihenkörpern über einem vollkommenen Körper der Charakteristik 2. Rev. Fac. Sci. Univ. Istanbul (A) **8**, 297–327 (1943). (German. Turkish summary) [MF 15200]

[Part I appeared in J. Reine Angew. Math. **183**, 148–167 (1941); these Rev. **4**, 237.] Let $k=\Omega\{t\}$ be the field of power series in t with coefficients in the perfect field Ω of characteristic 2. Suppose that ω is the natural valuation of k with $\omega(t)=1$, $\omega(\alpha)=0$ for $\alpha\neq 0$ in Ω. Let g be the valuation ring of ω. Then two equivalent quadratic forms over k are called arithmetically equivalent if the associated transformation matrix has coefficients in g. The author considers binary and ternary forms and exhibits complete systems for arithmetical equivalence. Let $y/\wp$ be a solution of the equation $x^2-x=y\epsilon k$. Then the study of the effect of the transformations (with coefficients in g) on a completely regular binary form $f=ax^2+bxy+cy^2$, $b\neq 0$, shows that the algebraic invariants (i) the field $K=k((ac/b^2)/\wp)$ and (ii) the algebra $C=(a, K)$ have to be supplemented by (i′) the order $\nu=-\omega(ac/b^2)$ and (iii) the multiplicative residue class $aE^2 \pmod{t^\mu}$, where E is the unit group of k, $\mu=\min[\omega(b/a), \omega(c/a)]$, and $\min[\omega(a), \omega(b), \omega(c)]$. Conversely, the data K, C, $aE^2 \pmod{t^\mu}$ with $C=(a, K)$, $\mu\equiv\nu \pmod 2$ for $\mu\leqq\nu$ always serve as a complete system of invariants for a class of completely regular forms. The representation theory by forms is set up. Moreover, the results are extended to quasilinear forms ax^2+cy^2. In the latter case considerable simplifications are possible since k has only one inseparable quadratic extension. The theory of ternary forms is reduced to the discussion of binary forms by observing that a ternary form F can always be expressed as $(ax^2+bxy+cy^2)+dz^2=f+ds^2$ after a suitable arithmetic equivalence transformation. An involved analysis leads to the fcllowing invariants for $F=f+s^2$: (iv) the multiplicative residue class $aE^2 \pmod{t^{M}}$, where $M=\min[\mu, v(a)-2\omega(a)]$ for $\omega(a)<0$, $M=0$ otherwise, with $v(a)$ the maximal x for which a is a quadratic residue mod t^x, μ for f as in [illegible]; (v) the additive residue class of ac/b^2 (mod $H(F)$), where (iii)

$$H(F)=\{a\lambda^2/b^2+c\kappa^2/b^2\} \pmod{\wp k}$$

for all κ, $\lambda\epsilon g$ and ν for f as in (i′); (vi) the algebra C for f as in (ii). Conversely, each set of data (iv) to (vi) belongs to a suitable form. *O. F. G. Schilling* (Chicago, Ill.).

Madhava Rao, B. S. Pauli's identities in the Dirac algebra. Proc. Indian Acad. Sci., Sect. A. **22**, 408–422 (1945). [MF 15679]

The author's summary is as follows. It is shown in this paper that by choosing suitable forms for 4×4 matrices as products of Dirac matrices and matrices of rank unity, and expressing them as linear combinations of the sixteen elements γ^A of the basis of the Dirac algebra, one can derive the generalized identities of Pauli which hold in this algebra. Generalizations are given for cases not dealt with by Pauli, and the use of his B-matrix is also avoided. The same method yields further "tensor," multilinear, and polynomial identities of which it is shown that the last two kinds of identities are derivable from bilinear and quadratic ones. It is pointed out that all types of identities can be deduced by considering five primitive types of matrices. *C. C. MacDuffee* (Madison, Wis.).

Schestakoff, W. Sur une calcul symbolique applicable à la théorie des schèmes électriques de relais. Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 45–48 (1944). (Russian. French summary) [MF 15193]

Elementary connections between the operation of "harmonic addition" $a\cdot b=(a^{-1}+b^{-1})^{-1}$, Boolean algebras, and the composition of electrical elements in parallel and in series. *H. Wallman* (Cambridge, Mass.).

Abstract Algebra

Rachevsky, P. Les problèmes les plus simples de "l'algèbre quasi-commutative" en connexion avec la théorie des valeurs caractéristiques des opérateurs différentiels. Rec. Math. [Mat. Sbornik] N.S. **10(52)**, 95–142 (1942). (French. Russian summary) [MF 12827]

[This is a continuation of a paper which appeared in the same Rec. N.S. **9(51)**, 511–544 (1941); these Rev. **3**, 265.] The author continues his investigations on noncommutative polynomials. One of the main problems of the present paper is the following. Let $P(x, y)$ be a second degree polynomial and $L(x, y)$ be linear. One wishes to determine when $P(x, y)$ is reducible with respect to $L(x, y)$, that is, when there exists a nonvanishing polynomial $Q(x, y)$ such that $P(x, y)Q(x, y)=0 \pmod{L(x, y)}$. A necessary and sufficient condition for this to happen is derived for so-called independent irreducibility. The method is based upon representations of the given polynomial in the form $P(x, y)=L_1L_2+\omega$, where L_1 and L_2 are linear and ω does not contain the variables.

The author then proceeds to an interpretation of the theory in terms of differential and difference operators. It is shown that the question of the integration of an operator of degree greater than 1 by means of linear operations is algebraically equivalent to the reducibility of P for a linear module L. Finally, it is shown that, if one considers all polynomials P which differ only in the constant term, there exist special values for which the polynomial becomes reducible. Examples show how these values correspond to the characteristic values in a number of operator problems. *O. Ore* (New Haven, Conn.).

McKinsey, J. C. C., and Tarski, Alfred. On closed elements in closure algebras. Ann. of Math. (2) **47**, 122–162 (1946). [MF 15662]

This is a continuation of the authors' paper in the same Ann. (2) **45**, 141–191 (1944); these Rev. **5**, 211. The authors first show that Brouwerian algebra (logic) is equationally definable and deduce various corollaries. They then observe that any "closure algebra" Γ becomes a Brouwerian algebra Γ^*, if $X\rightarrow Y$ is defined as the closure of $X \cap Y'$. Conversely, every Brouwerian algebra A is isomorphic with Γ^* for a minimal $\Gamma(A)$. Using these facts, results about closure algebras may be interpreted as results about Brouwerian algebra and the composite operations (or "algebraic functions") definable in the two types of algebra correspond. More significantly, the duality (between "open" and "closed" sets) which exists in closure algebras gives rise to a concealed duality in Brouwerian algebra. Decision procedures are found for Brouwerian equations. Free and "functionally free" Brouwerian algebras are constructed and discussed. The algebra of closed sets of any totally disconnected, normal, dense-in-itself topological space with a countable

basis is "absolutely free," in the sense that any relation involving Brouwerian operations which holds identically in it can be deduced from the postulates defining Brouwerian algebra. The "regular" elements of any Brouwerian algebra form a Boolean algebra. *G. Birkhoff.*

Tarski, Alfred. A remark on functionally free algebras. Ann. of Math. (2) 47, 163–165 (1946). [MF 15663]

It is shown that any "equationally definable" class of algebras consists precisely of the homomorphic images of subalgebras of direct powers of any "functionally free" member of the class, provided the latter exists.
G. Birkhoff (Cambridge, Mass.).

Birkhoff, Garrett. What is a lattice? Gaz. Mat., Lisboa 7, no. 27, 1–3 (1946). (Portuguese) [MF 15940]

Translation of an article in Amer. Math. Monthly 50, 484–487 (1943); these Rev. 5, 31.

Miyazaki, Sadataka. Über eine Menge, in der die Gleichartigkeit zwischen ihren Elementen definiert ist. I. Proc. Phys.-Math. Soc. Japan (3) 24, 197–207 (1942). [MF 15025]

The author calls a set with a quaternary relation (a, b, c, d) an "Art" (species) if (a, b, c, d) implies (c, d, a, b) and (a, b, d, c), that is, if it is symmetric on the unordered pairs (a, b) and (c, d). He discusses the properties possessed by such a relation in various examples, together with their logical dependence or independence, especially in the case when (a, b, c, d) means $a+b=c+d$ in an Abelian group. [Cf. J. Certaine, Bull. Amer. Math. Soc. 49, 869–877 (1943), in particular, § 5; these Rev. 5, 227.] *G. Birkhoff.*

Miyazaki, Sadataka. Über eine Menge, in der die Gleichartigkeit zwischen ihren Elementen definiert ist. II. Proc. Phys.-Math. Soc. Japan (3) 25, 101–115 (1943). [MF 15051]

[Cf. the preceding review.] The author develops further the theory of "species," including a Jordan-Hölder theorem, characterizations of the numbers of elements in finite "species" and "subspecies," and a proof that in certain cases the "subspecies" must form a complemented lattice. [Cf. O. Borůvka, Math. Ann. 118, 41–64 (1941); these Rev. 3, 200.] *G. Birkhoff* (Cambridge, Mass.).

Miyazaki, Sadataka. Verbandart. Proc. Phys.-Math. Soc. Japan (3) 25, 447–456 (1943). [MF 15063]
Miyazaki, Sadataka. Über Boolesche Algebra und Hauptideale im kleinen in der assoziativen Verbandart. Proc. Phys.-Math. Soc. Japan (3) 26, 43–46 (1944). [MF 15083]

A "lattice species" is a lattice which is a "species" (Art), in which (a, b, c, d) and (e, b, c, f) imply (d, e, a, f), and in which (a, b, c, d) and $a\geqq c$ imply $b\leqq d$. The author proves that this is an alternative definition of a lattice-ordered Abelian group, if we fix p and let $a+b=c$ mean (a, b, c, p); then (a, b, c, d) means $a+b=c+d$. He develops various properties of lattice-ordered Abelian groups in this notation and terminology. *G. Birkhoff* (Cambridge, Mass.).

Kawada, Yukiyosi. Über die Existenz der invarianten Integrale. Jap. J. Math. 19, 81–95 (1944). [MF 14997]

Let B be a Boolean algebra which is complete in the sense that, for any collection $\{a_\alpha\}$, $a_\alpha\epsilon B$, it is true that $\cup_\alpha a_\alpha\epsilon B$. Let $G=\{\sigma\}$ be a group of automorphisms of B such that

$$1^\sigma=1,\quad 0^\sigma=0,\quad (a\cap b)^\sigma=a^\sigma\cap b^\sigma,\quad (a\cup b)^\sigma=a^\sigma\cup b^\sigma.$$

A measure μ (numerically valued, nonnegative, countably additive) defined on B such that either $\mu(1)<\infty$ or $1=\cup_n a_n$, $\mu(a_n)<\infty$ $(n=1, 2, \cdots)$, is said to be invariant if $\mu(a^\sigma)=\mu(a)$ for $a\epsilon B$, $\sigma\epsilon G$. The measure μ is said to be proper if $\mu(a)>0$ implies $a>0$. Following E. Hopf [Trans. Amer. Math. Soc. 34, 373–393 (1932)], the author defines a and b of B to be equivalent by division (symbolically, $a\sim b$) if

$$\begin{aligned} &a=\cup_n a_n,\quad b=\cup_n b_n, && a_n, b_n\epsilon B,\ n=1, 2, \cdots,\\ &a_i\cap a_j=b_i\cap b_j=\phi, && i\neq j,\\ &a^{\sigma_n}=b_n,\quad \sigma_n\epsilon G, && n=1, 2, \cdots. \end{aligned}$$

An element a of B is said to be finite if there exists no element b of B such that $b<a$ and $b\sim a$. Let Z denote the set of all invariant elements of B, that is, $c\epsilon B$ if and only if $c^\sigma=c$ for all σ in G. It follows that Z is a complete Boolean subalgebra of B. Let $e(a)$ be the set generated by a, that is, $e(a)=\cup_\sigma a^\sigma$, where σ ranges over all of G.

The principal result of the paper is the following theorem. Let B be a complete Boolean algebra such that corresponding to any set $\{a_\alpha\}$ $(a_\alpha\epsilon B)$ there exists a countable subset $\{a_{\alpha_n}\}$ such that $\cup_\alpha a_\alpha=\cup_n a_{\alpha_n}$. Let G be a group of automorphisms of B and suppose that there exists a finite element a^* such that $e(a^*)=1$. Then, corresponding to any measure μ_0 defined on Z with $\mu_0(1)=1$, there exists an invariant measure defined on B such that $\mu(a^*\cap c)=\mu_0(c)$ if $c\epsilon Z$, $\mu(b)>0$ if $\mu_0(e(b))>0$. If μ_0 is proper on Z, μ is proper on B. This result includes that of Hopf. The methods are algebraic, and, as the author remarks, are similar to those employed by J. von Neumann ["Continuous Geometry," Institute for Advanced Study, Princeton, N. J., 1936, 1937].
G. A. Hedlund (Charlottesville, Va.).

Foster, Alfred L. The theory of Boolean-like rings. Trans. Amer. Math. Soc. 59, 166–187 (1946). [MF 15321]

A "Boolean-like" ring is a commutative ring H with unit element such that $a+a=0$ for all a in H and $ab(a+b+ab)=ab$ for any two elements a, b of H. The main properties of such rings are (1) $a^4=a^2$ for all a in H; (2) the nilpotent elements of H form an ideal N such that the product of two elements of N is 0; (3) the idempotent elements of H form a Boolean subring J of H and every element of H can be expressed in one and only one way as a sum of an idempotent and of a nilpotent element. Properties (2) and (3), together with the fact that the ring is commutative, has a unit element and is of characteristic 2, characterize Boolean-like rings. The structure of the subrings J and N being well-known, the study of the structure of H reduces to that of the law of multiplication of an element of J by an element of N; that in turn amounts to the study of the representations of the ring J into the ring of endomorphisms of the additive group N, a study which the author intends to treat in a further paper. Throughout the paper, the author translates most of the results into the language of "logical operations" on H, defined by $a\Delta b=a+b-ab$, $a\oplus b=a+b-1$, $a^*=1-a$. *J. Dieudonné* (São Paulo).

Matusita, Kameo. Über ein bewertungstheoretisches Axiomensystem für die Dedekind-Noethersche Idealtheorie. Jap. J. Math. 19, 97–110 (1944). [MF 14998]

A valuation-theoretic criterion is given for an integral domain $\mathfrak{o}$ to have the classical Dedekind-Noether ideal theory, that is, for every proper ideal of $\mathfrak{o}$ to be uniquely expressible as a product of maximal prime ideals. A necessary and sufficient condition is: (1) $\mathfrak{o}$ is the intersection of a set of discrete valuation rings; (2) each element of $\mathfrak{o}$ is a unit in all but a finite number of the valuation rings;

(3) given any two of the valuation rings, there is an element e in $\mathfrak{o}$ such that e is a non-unit in one and $1-e$ is a non-unit in the other. This criterion is applied to give alternate proofs of several known theorems. It is also shown that, if every proper ideal in $\mathfrak{o}$ is the product of prime ideals, then the primes must be maximal and the decomposition unique.

I. S. Cohen (Cambridge, Mass.).

Brown, Bailey, and McCoy, Neal H. Rings with unit element which contain a given ring. Duke Math. J. 13, 9–20 (1946). [MF 15870]

Let D be an arbitrary set of rings with unit element containing the given ring $\mathfrak{R}$. A subset S of D is called a complete set of extensions of $\mathfrak{R}$ in D if every ring in D contains a subring isomorphic over $\mathfrak{R}$ to a ring in S. A construction is given for a complete set of extensions in the case where D consists of all rings with unity containing $\mathfrak{R}$ and also in the case where the rings in D are restricted to have the same characteristic as $\mathfrak{R}$. In these two cases a criterion is given for a complete set of extensions to exist consisting of a single ring. For example, the condition in the first case is that the mode of $\mathfrak{R}$ is 0 or 1. The mode, a fundamental concept in this paper, is defined as the nonnegative generator of the ideal of all integers α for which there exists an element $a \epsilon \mathfrak{R}$ such that $\alpha r = ar = ra$ for all $r \epsilon \mathfrak{R}$. There are various other results. *I. S. Cohen* (Cambridge, Mass.).

Nakayama, Tadasi. On Frobeniusean algebras. III. Jap. J. Math. 18, 49–65 (1942). [MF 14967]

The author has studied Frobeniusean algebras and rings in a number of previous papers [Ann. of Math. (2) 40, 611–633 (1939); 42, 1–21 (1941); Proc. Imp. Acad. Tokyo 17, 53–56 (1941); these Rev. 1, 3; 2, 344]. The last of these papers contains a short account of some of the new results whose proof is given in the present paper. Let A be a ring which possesses a unit element and which satisfies the minimum condition for one-sided ideals. It is shown that, if the lattices Λ_0 and P_0 of the completely reducible left and right ideals in A are dual-isomorphic, respectively, to the lattices P_1 and Λ_1 of the right and left ideals of A containing the radical, then A is quasi-Frobeniusean. If the dual-isomorphism $\mathfrak{l} \rightarrow \mathfrak{r}$ between Λ_0 and P_1 can be chosen such that the $\mathfrak{l}$-dimension $d_l(\mathfrak{l})$ of $\mathfrak{l}$ is always equal to the r-dimension $d_r(A/\mathfrak{r})$, then A is a Frobeniusean ring. It follows as a corollary that, if the left and right ideal lattices Λ and P of a ring A are dual-isomorphic, then A is quasi-Frobeniusean; if for at least one dual-isomorphism $\mathfrak{l} \rightarrow \mathfrak{r}$ the condition $d_l(\mathfrak{l}) = d_r(A/\mathfrak{r})$ is satisfied, then A is Frobeniusean. In the case of algebras, the condition for the dimensions can be replaced by the relation $(\mathfrak{l}:F)+(\mathfrak{r}:F)=(A/F)$, where F is the ground field. A generalization of these results is given in which two algebras A and A' are considered. If Λ_0', P_0', Λ_1' and P_1' have the same significance for A' as Λ_0, P_0, Λ_1, P_1 have for A, and if Λ_0, P_0, Λ_1, P_1 are dual-isomorphic with P_1', Λ_1', P_0', Λ_0', respectively, then both A and A' are quasi-Frobeniusean. An additional assumption concerning the dimensions implies again that A is Frobeniusean. Some interesting examples are discussed. *R. Brauer* (Toronto, Ont.).

Abe, Makoto. Irreduzibilität und absolute Irreduzibilität der Matrizensysteme. Proc. Phys.-Math. Soc. Japan (3) 24, 769–789 (1942). [MF 15042]

Let S be an algebraic system, such as a group, an associative algebra or a Lie algebra. Then we may consider representations $x \rightarrow R(x)$ of S by matrices $R(x)$ over a field K, the representations having the usual properties. Let K have degree r over a subfield F so that the right multiplications of K form a field K' over F consisting of r-rowed square matrices over F. Then every t-rowed representation $R(x)$ over K of x in S may be replaced by a derived rt-rowed representation $R'(x)$ obtained by replacing each element of $R(x)$ by the corresponding element of K'. The author shows that if $x \rightarrow R_1(x)$ is an irreducible representation of S in F then there exists an overfield K of finite degree over F and an absolutely irreducible representation $x \rightarrow R(x)$ over K of S, such that $x \rightarrow R_1(x)$ and $x \rightarrow R'(x)$ are equivalent. A partial converse is also obtained. The paper closes with considerations of a number of special cases and with a result similar to that above for the case where K is a division algebra over F. *A. A. Albert* (Chicago, Ill.).

Lee, H. C. On Clifford's algebra. J. London Math. Soc. 20, 27–32 (1945). [MF 15931]

Clifford's algebra over an algebraically closed field F of characteristic not two is the associative algebra of order 2^n generated by n elements u_i such that $u_i^2=1$, $u_iu_j=-u_ju_i$ for $i \neq j$ and $i, j=1, \cdots, n$. The author gives what he calls a modern proof of the result stating that this algebra is semisimple by a trace argument involving the regular representation by right multiplications. He shows that the center has order one if n is even and order two if n is odd. The number of inequivalent irreducible representations is also determined. The algebra is actually a direct product M of total matric algebras of degree two when n is even and the direct product of M by a commutative semisimple algebra of order two when n is odd. It seems to the reviewer that the demonstration of these direct product relations, which reduces the theorems of the paper to standard results, would be the most elegant modern demonstration of them.

A. A. Albert (Chicago, Ill.).

★**Weyl, Hermann. Fundamental domains for lattice groups in division algebras. I.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 218–232, Füssli, Zürich, 1945.

Weyl, Hermann. Fundamental domains for lattice groups in division algebras. II. Comment. Math. Helv. 17, 283–306 (1945).

According to the author, these papers were conceived as a commentary on the "all too laconic" part III of C. L. Siegel's paper [Ann. of Math. (2) 44, 674–689 (1943); these Rev. 5, 228], with the aim of making Siegel's results concerning the group of units in a simple order more accessible to the nonspecialist. Consequently, the account of the Minkowski-Weyl-Siegel methods of reduction of positive quadratic forms contained in these papers has been made reasonably self-contained. The first two sections of the first paper contain pertinent background material on matrices, linear algebras, vectors and lattices. The next and final two sections introduce and discuss the properties of a conjugate $\bar{\alpha}$ of an element α in a semi-simple algebra F_K over the field K of all real numbers, and that of a "quadratic" form $\Gamma[\xi]=\bar{\xi}'\Gamma\xi=\sum_{\mu,\nu}\bar{\xi}_\mu\gamma_{\mu\nu}\xi_\nu$, $\bar{\Gamma}=\Gamma'$, in F_K. This quadratic form is said to be positive if the associated real quadratic form $G[x]$ is positive. Here $G=[G_{ij}]$ is the real symmetric compound matrix whose elements G_{ij} are matrices representing γ_{ij} in terms of the normal basis. The trace of γ_{mm} is the trace of G_{mm}. The trace $t_\Gamma = \mathrm{tr}\,(\Gamma[\xi]) = \sum_{\mu,\nu} \mathrm{tr}\,(\bar{\xi}_\mu\gamma_{\mu\nu}\xi_\nu)$ of

the quadratic form $\Gamma[\xi]$ is an ordinary positive quadratic form in the real components of the variables ξ_ν. It is upon this form t_Γ [cf. Siegel, loc. cit., p. 684] and not upon $G[x]$ that the reduction procedures in the second paper are based. The first part of this paper contains an adaptation to positive quadratic forms in F_K of the Minkowski-Weyl [Weyl, Trans. Amer. Math. Soc. **48**, 126–164 (1940); **51**, 203–231 (1942); these Rev. **2**, 35; **3**, 272] geometrical method of reduction depending essentially upon a generalization of Minkowski's fundamental inequality [Geometrie der Zahlen, Teubner, Leipzig, 1910, pp. 196–199]. The last part is devoted to a discussion of the more general approach of Siegel [loc. cit.; Abh. Math. Sem. Hansischen Univ. **13**, 209–239 (1940); these Rev. **2**, 148] and to a comparison of the two methods. *A. E. Ross* (St. Louis, Mo.).

Malcev, A. Commutative subalgebras of semi-simple Lie algebras. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 291–300 (1945). (Russian. English summary) [MF 15359]

This paper is devoted to the determination of the commutative subgroups of maximum dimension of the semisimple complex Lie groups. The problem is equivalent to that of determining the commutative subalgebras of maximum dimensionality of the complex semisimple Lie algebras. In the latter form the problem is reduced to that of determining the commutative subalgebras of maximum dimensionality containing only nilpotent elements of the simple Lie algebras. The result obtained is the following theorem. Every simple Lie algebra over the complex field with the exception of B_4, D_4 and G_2 possesses only one class of conjugate commutative subalgebras of maximum dimensionality. The order is $[n^2/4]$ for A_{n-1} $(n>3)$, $\frac{1}{2}n(n-1)+1$ for B_n $(n>4)$, $\frac{1}{2}n(n+1)$ for C_n, $\frac{1}{2}n(n-1)$ for D_n $(n>4)$, 16, 27, 36, 9, 5, respectively, for E_6, E_7, E_8, F_4, B_3. The algebra B_4 has two conjugate classes of commutative subalgebras of maximum order 7, D_4 has two classes of maximum order 6 and G_2 has three classes of order 3. The results for A_{n-1} are well-known results of Schur; a simplified derivation was given by the reviewer [Bull. Amer. Math. Soc. **50**, 431–436 (1944); these Rev. **6**, 33]. *N. Jacobson.*

Tôyama, Hiraku. Über die Darstellungsklasse der Fundamentalgruppe. Proc. Phys.-Math. Soc. Japan (3) **26**, 41–42 (1944). [MF 15082]

The author shows that the classes of unitary representations [in the sense of A. Weil, J. Math. Pures Appl. (9) **17**, 47–87 (1938)] of the fundamental group of an algebraic function field form a bounded set. *O. F. G. Schilling.*

Deuring, Max. Reduktion algebraischer Funktionenkörper nach Primdivisoren des Konstantenkörpers. Math. Z. **47**, 643–654 (1942). [MF 15907]

The author discusses an arithmetic version of the principle of degeneration for fields of algebraic functions of one variable $K=k(x, y)$. Suppose that k is algebraically closed in K. Let $\mathfrak{p}$ be a discrete rank one valuation on k with the residue class field $\bar{k}$. Then $\mathfrak{p}$ has a prolongation $\mathfrak{p}_x$ to $k(x)$, where

$$\mathfrak{p}_x[(a_0+\cdots+a_rx^r)/(b_0+\cdots+b_sx^s)] = \min_i[\mathfrak{p}(a_i)]-\min_i[\mathfrak{p}(b_i)].$$

Then the residue class $\bar{x}$ of x is transcendental over $\bar{k}$ and $k(x)$ modulo $\mathfrak{p}_x=\bar{k}(\bar{x})$. Assume now that $\mathfrak{p}_x$ remains completely inert in K. Then the following lemma holds for the reduction of a finite dimensional k-submodule M of K. Let $\bar{M}$ be the $\bar{k}$-module of all residues $\bar{m}$ mod $\mathfrak{p}_x$, $m\in M$, $\mathfrak{p}_x(M)=0$; then $[M:k]=[M \bmod \mathfrak{p}_x:\bar{k}]$. The proof of this lemma uses the imbedding of K in its $\mathfrak{p}_x$-adic closure. Suppose now that $\bar{k}$ is the coefficient field of $\bar{K}$. Then the lemma, with an investigation of the multipla of the denominator of x, implies that the genus of $\bar{K}$ is bounded by the genus of K. Finally, if the genus is preserved then a mapping $\mathfrak{a}\to\bar{\mathfrak{a}}$ of the divisors $\mathfrak{a}$ of K on the divisors $\bar{\mathfrak{a}}$ of $\bar{K}$ can be defined which (i) is a multiplicative homomorphism, (ii) preserves the degree of a divisor and (iii) maps each principal divisor (a) with $\mathfrak{p}_x(a)=0$ on the principal $(\bar{a})$. These properties imply in turn that the dimension of a divisor class of sufficiently large degree remains unchanged for the mapping $K\to\bar{K}$. In conclusion, it is shown that all but a finite number of valuations $\mathfrak{p}$ of k satisfy the hypotheses leading to the preceding results. *O. F. G. Schilling* (Chicago, Ill.).

Inaba, Eizi. Algebraische Funktionenkörper und Algebren mit allgemeinem Koeffizientenkörper. I. Jap. J. Math. **18**, 635–662 (1943). [MF 14977]

Let K be an algebraic function field obtained from a field of constants Λ by a simple transcendental extension $\Lambda(z)$ followed by a finite algebraic extension $\Lambda(z, y_1, \cdots, y_r)$, where each y_j satisfies an irreducible equation $F_j(z)$ over $\Lambda(z)$; K can then be regarded as an algebra over $\Lambda(z)$. A coefficient extension of K is obtained by replacing Λ by a larger field of constants Λ'; K is extended as an algebra to K', and may no longer be a field. A coefficient contraction arises if Λ is replaced by a smaller field Λ_0 which contains all coefficients appearing in the equations $F_j(t)$. Finally, if Λ is a field of algebraic numbers or is itself an algebraic function field, each natural discrete one-dimensional valuation V of Λ provides a homomorphism of Λ on the residue class field Λ^* plus ∞, and hence a homomorphism of $\Lambda(z)$ upon $\Lambda^*(z)$ plus ∞. The algebra $K^*=\Lambda^*(z, y_1, \cdots, y_r)$ obtained by applying this homomorphism to the coefficients of the defining equations F_j is then said to be obtained by a Restbildung. The effect of these three operations on the genus, the prime divisors, the principal orders, the normal bases and the divisor classes is exhaustively investigated.

In a proof of the Hilbert irreducibility theorem, Eichler [Math. Ann. **116**, 742–748 (1939)] has shown that a polynomial $f(x, y)$, absolutely irreducible over an algebraic number field k, remains absolutely irreducible when its coefficients are reduced modulo any but a finite number of prime divisors p of k. Generalizing this result, the author proves that the algebra K^* described above is an algebraic function field over Λ^* except for a finite number of Restbildungen. Unfortunately, the statement of the theorem does not properly restrict the Restbildungen considered to the specific type described at the foot of page 641. There are other minor inaccuracies; for example, the isomorphism of page 638, line 10, must be the natural isomorphism.

These methods are used to prove that, if K has genus g over an algebraically complete coefficient field Λ of characteristic ∞, then the number of divisor classes with mth power principal is m^{2g}. This is a classical consequence of Abel's theorem if Λ is the field of complex numbers and the general case is reduced to the classical case by the operations cited. The same result has been obtained by similar methods by O. F. G. Schilling [Amer. J. Math. **61**, 59–80 (1939), in particular, p. 76]. *S. MacLane.*

Gaeta, Federico. Note.—An application of linear algebra to the theory of the simple algebraic extensions of a field. Revista Mat. Hisp.-Amer. (4) 5, 251–254 (1945). (Spanish) [MF 15885]

A simple algebraic extension of a field K, obtained by adjoining to K a root of a polynomial $\phi(x)$ irreducible in $K[x]$, is represented by $K[X]$, where X is a matrix with elements in K and whose minimum function is $\phi(x)$; X may be taken in K_n, the ring of matrices of order n (the degree of $\phi(x)$) over K. If Δ is a normal separable extension of K, the Galois group of Δ over K is represented by a group of inner automorphisms of K_n. If G is a group of order n and A_i $(i=1, \cdots, n)$ the members of the regular representation of G in K_n, then G is isomorphic to the group of inner automorphisms $X \rightarrow A_i^{-1}XA_i$ $(X \epsilon K)$. A field contained in K_n and of degree n over K which is invariant under each of these automorphisms is normal over K and has G for its Galois group relative to K. *J. L. Dorroh.*

Moriya, Mikao. Einige Eigenschaften der endlichen separablen algebraischen Erweiterungen über perfekten Körpern. Proc. Imp. Acad. Tokyo 17, 405–410 (1941). [MF 14720]

Moriya, Mikao. Struktur der Divisionsalgebren über diskret bewerteten perfekten Körpern. Proc. Imp. Acad. Tokyo 18, 5–11 (1942). [MF 14735]

Moriya, Mikao. Algebrenklassengruppen über diskret perfekten Körpern. Proc. Imp. Acad. Tokyo 18, 37–38 (1942). [MF 14738]

Moriya, Mikao. Die Theorie der Klassenkörper im Kleinen über diskret perfekten Körpern. I. Proc. Imp. Acad. Tokyo 18, 39–44 (1942). [MF 14739]

Moriya, Mikao. Die Theorie der Klassenkörper im Kleinen über diskret perfekten Körpern. II. Proc. Imp. Acad. Tokyo 18, 452–459 (1942). [MF 14772]

Nakayama, Tadasi, und Moriya, Mikao. Zur Theorie der Normenrestsymbole über diskret perfekten Körpern. Proc. Imp. Acad. Tokyo 19, 129–131 (1943). [MF 14803]

Nakayama, Tadasi, und Moriya, Mikao. Die Theorie der Klassenkörper im Kleinen über diskret perfekten Körpern. III. Proc. Imp. Acad. Tokyo 19, 132–137 (1943). [MF 14804]

Local class field theory (that is, the description of all the Abelian extension fields of a $\mathfrak{p}$-adically closed field k in terms of multiplicative groups of elements of k) is well known for two classes of fields: the $\mathfrak{p}$-adic closures of the rational field and of function fields in one variable over Galois fields, and their algebraic extensions. The classical proofs make considerable use of the special properties of these fields. This series of papers develops local class field theory for all fields k which are complete under a discrete valuation and whose residue class field $\mathfrak{k}$ (that is, k-elements with value at most 1 modulo elements with value less than 1) satisfies the axioms (1) $\mathfrak{k}$ has no inseparable extensions; (2) for every n, $\mathfrak{k}$ has exactly one extension of degree n. These axioms hold in the classical case, since $\mathfrak{k}$ is then a Galois field. From the axioms it follows [second paper] that the extension fields of $\mathfrak{k}$ are all cyclic and that every element of $\mathfrak{k}$ is norm from any extension field. The field k has exactly one unramified extension field of degree n, and it is cyclic.

The theory of cyclic algebras is essentially the same as in the classical case, and yields the theorem that every cyclic field has norm group isomorphic to its Galois group. This is proved and extended to Abelian fields in the fourth paper. The fifth contains proofs that every class field is Abelian and that the largest Abelian subfield of any separable extension field is the class field for its norm group (limitation theorem).

The sixth paper (which requires only the first two as prerequisites) defines the norm residue symbol by use of cyclic algebras. Since in this general case there is no absolute norm, and hence no Artin symbol, it is necessary to choose (arbitrarily) a generating automorphism S for the union field of all the unramified extension fields of k. This S exists because the field in question is ideal-cyclic in the sense of Krull [Math. Ann. 100, 687–698 (1928)]. Invariant of an algebra and norm residue symbol can then be introduced as usual, using S in place of the Artin symbol. This norm residue symbol could be used to give another proof of the isomorphism of Galois group to norm group.

Only with the existence theorem does this theory differ from the classical one. The seventh paper proves that every closed number group of finite index has a class field. But the topology used is obtained by using the norm residue symbol described above to impose Krull's topology of the Galois group [see the reference above] on the group of elements of k. Thus closure is defined in terms of the extension fields, rather than by arithmetic properties of the field k itself. The group of nth powers of k-elements is shown to be always closed; from this it follows easily that in the classical case every subgroup of finite index has a class field. An example is promised which will show that the closure condition is sometimes necessary. *G. Whaples.*

Nakayama, Tadasi. A theorem on the norm group of a finite extension field. Jap. J. Math. 18, 877–885 (1943). [MF 14987]

One of the principal difficulties of the generalized local class field theory of Moriya and Nakayama [see the preceding review] is the proof of the limitation theorem. The classical proof will not work, since it uses the existence theorem. This paper contains a new and elegant proof based on the theory of simple algebras. The principal tool is the following mapping, due to Nakayama [Math. Ann. 112, 85–91 (1935)] and Akizuki [Math. Ann. 112, 566–571 (1936)]. Let $(a)=a_{S,T}$ be a factor set for the normal field Ω/k and define $F(R; (a))=\prod_S a_{R,S}$, where S runs through the Galois group. Then $R\rightarrow F(R, (a))$ gives a homomorphic mapping of the Galois group into the norm class group of k. The elements of the commutator subgroup go into the unit class. Assuming certain properties of the group of algebras over k (which hold for the fields of generalized local class field theory) it is shown that, if L/k contains the greatest Abelian subfield of Ω/k, then every $N_{L/k}$ is an $N_{\Omega/k}$; from this the limitation theorem follows. *G. Whaples.*

Krasner, Marc. Une généralisation de la notion de corps—corpoïde. Un corpoïde remarquable de la théorie des corps valués. C. R. Acad. Sci. Paris 219, 345–347 (1944). [MF 15263]

A set Q with two operations, addition and multiplication, is called a corpoid if the following axioms are satisfied: (1) Q is the union of a multiplicative group Q^* (unit element 1) with a zero element 0 for which $a\cdot 0=0\cdot a=0$ for all a in Q; (2) addition is defined only for certain couples a, b of Q, of which it is said that a is addible with b (notation: $aADb$); $0ADb$ for all b in Q; the set R of all elements addible with 1 forms an Abelian group with respect to addition; (3) if $aADb$ and $c\epsilon Q$ then $acADbc$, $caADcb$ and $(a+b)c$

$=ac+bc$ and $c(a+b)=ca+cb$. The set R of elements addible with 1 forms a (not necessarily commutative) field, and is called the field of the corpoid Q. The group R^* of nonzero elements of R is a normal subgroup of Q^* and Q^*/R^* is called the group of Q. Commutative extension corpoids are discussed and Galois extensions defined. Definitions of the Galois group and Galois hypergroup may then be given as for commutative fields. [See M. Krasner, Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4°. (2) 11, no. 8 (1937)]. The Galois correspondence between separable extension subcorpoids and subhypergroups of the Galois hypergroup then follows.

Let K be a field with a valuation and $\mathfrak{P}$ its prime ideal. Let the elements of K be divided into classes so that α and β belong to the same class if and only if $\alpha/\beta-1\epsilon\mathfrak{P}$. All elements α in a class a have the same valuation $|a|$. All elements $-\alpha$ form a class $-a$. The product of two classes a, b is the class consisting of all products $\alpha\beta$, $\alpha\epsilon a$, $\beta\epsilon b$. Two classes a, b are addible if and only if $|a|=|b|$ or $|a|=0$. Then the class $a+b$ consists of all sums $\alpha+\beta$. The set S of all classes is a commutative corpoid and is called the skeleton of K. Its field R and group M can be identified with the field of residues and the valuation module of K.

D. C. Murdoch (Vancouver, B. C.).

Krasner, Marc. Nombres semi-réels et espaces ultramétriques. C. R. Acad. Sci. Paris **219**, 433–435 (1944). [MF 15275]

Let ρ_0 be any real number and ζ one of the three signs $-, n, +$. A semi-real number is a couple of the form (ρ_0, ζ). The semi-real numbers are ordered as follows: $(\rho_0, \zeta)<(\rho_0', \zeta')$ if $\rho_0<\rho_0'$, while $(\rho_0, -)<(\rho_0, n)<(\rho_0, +)$. If $\rho=(\rho_0, \zeta)$ is a semi-real number, $\rho_0=[\rho]$ is called its real value and $\zeta=\langle\rho\rangle$ is called its type (espèce). Two semi-real numbers ρ and ρ' are of compatible (non-opposées) types if $\langle\rho\rangle=\langle\rho'\rangle$ or if either $\langle\rho\rangle=n$ or $\langle\rho'\rangle=n$. The sum and product of two semi-real numbers ρ, ρ' of compatible type are defined by $[\rho+\rho']=[\rho]+[\rho']$, $[\rho\cdot\rho']=[\rho]\cdot[\rho']$ and $\langle\rho+\rho'\rangle=\langle\rho\cdot\rho'\rangle=i(\langle\rho\rangle, \langle\rho'\rangle)$, where $i(+, +)=i(+, n)=i(n, +)=+$, $i(n, n)=n$, and $i(n, -)=i(-, n)=i(-, -)=-$. The mapping $(\rho_0, n)\rightarrow\rho_0$ is an isomorphism which preserves order and hence (ρ_0, n) can be identified with the real number ρ_0.

A metric space E is said to be ultrametric if the metric $d(a, b)$ satisfies the condition $d(a, c)\leqq\max[d(a, b), d(b, c)]$ for all a, b, c in E (for example, a field with non-Archimedean valuation, where $d(a, b)=|a-b|$). If $\rho_0>0$ is a semi-real number with $\langle\rho\rangle\neq-$, and if $a\epsilon E$, the set $C_E(a;\rho)$ consists of all elements e of E such that $d(a, e)<\rho$. This set is called a circle of radius ρ and centre a, and is of the first or second kind according as $\langle\rho\rangle=+$ or n. Two circles are either disjoint or one is contained in the other. Every point in a circle is a centre. Every open set contained in E is a sum of disjoint circles.

Let P be an equivalence relation in E such that if $a\equiv b$, (P) and $d(a', b')\leqq d(a, b)$ then $a'\equiv b'$, (P). Then P is called a divisor of E. Every divisor corresponds to a division P_ρ of E into circles of radius ρ and conversely. The least ρ for which $P=P_\rho$ is called the valuation of P and denoted by $|P|$, while $\omega(P)=-\log|P|$ is the order of P. Also, P is of the first or second kind according as $\langle|P|\rangle=+$ or n. The product $P\cdot P'$ is P_ρ, where $\rho=|P|\cdot|P'|$ and P divides P' if $|P|\geqq|P'|$. The quotient space E/P is defined. If E is a field with a valuation its divisors may be identified with its ideals and an ideal may therefore be characterized by a single semi-real number. *D. C. Murdoch.*

Krasner, Marc. Hypergroupes extramoduliformes et moduliformes. C. R. Acad. Sci. Paris **219**, 473–476 (1944). [MF 15278]

Krasner, Marc. Théorie de la ramification dans les extensions finies des corps valués: hypergroupe de décomposition. C. R. Acad. Sci. Paris **219**, 539–541 (1944). [MF 15285]

Krasner, Marc. Théorie de la ramification dans les extensions finies des corps valués: hypergroupes d'inertie et de ramification; théorie extrinsèque de la ramification. C. R. Acad. Sci. Paris **220**, 28–30 (1945). [MF 13478]

Krasner, Marc. Théorie de la ramification dans les extensions finies des corps valués: différente et discriminant; théorie intrinsèque de la ramification. C. R. Acad. Sci. Paris **220**, 761–763 (1945). [MF 15234]

Krasner, Marc. Théorie de la ramification dans les extensions finies des corps valués: compléments et applications. C. R. Acad. Sci. Paris **221**, 737–739 (1945). [MF 15370]

In these notes the author applies the results announced in the two notes reviewed above, together with his theory of hypergroups [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4°. (2) 11, no. 8 (1937)] to the theory of ramification in finite extensions of fields with valuations. A corpoid Q [see the preceding reviews] is without torsion if its group G has no elements of finite order. Let Q be a commutative corpoidal extension of q, without torsion, and let Q'/q be a Galois extension of Q/q. Let K', K and k be the fields of Q', Q, and q, and $g_{Q/q}$ the Galois hypergroup of Q/q. The character hypergroup $\mathfrak{X}_{Q/q}$ is the set of all characters, or mappings of the form $\chi_\sigma=a\rightarrow\sigma a/a$, $a\epsilon Q^*$ and $\sigma\epsilon g_{Q/q}$, with a suitable law of combination that ensures that $\chi_\sigma\rightarrow\sigma$ is an isomorphism of $\mathfrak{X}_{Q/q}$ and $g_{Q/q}$. The set $\mathfrak{X}_{G/g}$ of $\chi\epsilon\mathfrak{X}_{Q/q}$ such that, for $a\epsilon K^*$, $\chi(a)=1$, is the set of ordinary characters of G/g in K'. If Q/q is Galois, $g_{Q/q}$ is a group and is the direct product $g_{K/k}\times\mathfrak{X}_{G/g}$. The corpoid Q_T corresponding under the Galois theory to $\mathfrak{X}_{G/g}$ is called the corpoid of inertia of Q/q.

Let K^*/k be a Galois extension of K/k and let $|\cdots|$ and $|\cdots|^*$ be valuations of K and K^*, respectively. An isomorphism σ of K/k in K^* is metric if, for all α in K, $|\sigma\alpha|^*=|\alpha|^*=|\alpha|$. The set $Z(K/k; |\cdots|^*)$ of metric isomorphisms of K/k in K^*, under a suitable definition of product, is a hypergroup$_D$, and is called the hypergroup of decomposition of K/k for the valuation $|\cdots|^*$. If $\sigma_1, \sigma_2\epsilon Z$ their product in Z is not necessarily identical with their product in $g_{K/k}$. A property of "stability" is defined for subhypergroups A of Z which provides a necessary and sufficient condition that they be subhypergroups of $g_{K/k}$, and hence that the corresponding extension fields K_A should exist. If K/k is Galois or if k is complete, a field K_A will correspond to every subhypergroup A of Z. In particular, the field $K_{Z/k}$ corresponding to Z is called the decomposition field of K/k. Its skeleton S_Z is identical with the skeleton s of k.

An element σ of $Z(K/k; |\cdots|^*)$ induces an isomorphism $\bar\sigma$ of the skeleton S/s in S^*, where s, S and S^* are the skeletons of k, K and K^*. The mapping $\sigma\rightarrow\bar\sigma$ is a normal homomorphism of $Z_{K/k}$ on $g_{S/s}$. The set $V_{K/k}(|\cdots|^*)$ of elements σ which are mapped in this homomorphism on the identical isomorphism of S is called the hypergroup of ramification of K/k; it is a normal subhypergroup of $Z(K/k; |\cdots|^*)$. Also $Z/V\cong g_{S/s}\cong\mathfrak{X}_{S/s}$. The inertial hypergroup is defined as the set of all σ in $Z(K/k; |\cdots|^*)$ for which $\bar\sigma\epsilon T_{S/s}$, the inertial

group of S/s. The higher ramification groups and their corresponding fields are also defined. Several further extensions and applications are made, including a discussion of discriminants and differents. *D. C. Murdoch.*

NUMBER THEORY

van der Pol, Balth. Music and elementary theory of numbers. Music Review 7, 1–25 (1946). [MF 16322]

Dyson, F. J. A theorem on the densities of sets of integers. J. London Math. Soc. **20**, 8–14 (1945). [MF 15928]

Let $A_1, \cdots, A_n$ be sets of nonnegative integers each containing the number 0. Let $A(m)$ denote the number of positive integers in A which are not larger than m. A sum of rank r is a sum of the form $A_{i_1}+\cdots+A_{i_r}$, where $+$ denotes the usual addition of number sets. Put $\phi_r(m)=\sum_s s(m)$, where $s=A_{i_1}+\cdots+A_{i_r}$ and the sum extends over all combinations $i_1, \cdots, i_r$ of the numbers $1, 2, \cdots, n$. Let g be an integer and γ a real number such that (*) $\phi_1(m)\geqq\gamma m$ for $m=1, 2, \cdots, g$. Put $\delta=\min(1, \gamma)$. The author proves the inequality $\phi_r(m)\geqq\binom{n-1}{r-1}\delta m$, $m=1, \cdots, g$; $r=1, \cdots, n$.

For $r=n$ one obtains a generalization to n summands of a theorem first proved by the reviewer [Ann. of Math. (2) **43**, 523–527 (1942); these Rev. **4**, 35] for 2 summands. It seems worth remarking that in the reviewer's paper the values of m in the inequality (*) were restricted to values not in $A_1+\cdots+A_n$. *H. B. Mann* (Columbus, Ohio).

Simmons, H. A. Note on use of matrices in solving linear Diophantine equations. Tôhoku Math. J. **48**, 71–74 (1941). [MF 16350]

The Diophantine equation $\sum_{i=1}^{n} a_i x_i = k$ has been solved by a method of substitutions [cf. É. Cahen, Théorie des Nombres, vol. 1, Hermann, Paris, 1914, chaps. 9, 10] and by various methods employing continued fractions. The writer outlines the method of substitutions using matrix notation, so that "one can see at a glance the nature and number of non-singular linear transformations that one uses in solving a general equation." *I. Niven.*

Vandiver, H. S. On the number of solutions of some general types of equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. **32**, 47–52 (1946). [MF 15516]

The author considers the equation

$$c_1x^{a_1}+\cdots+c_sx^{a_s}+c_{s+1}=0,$$

where the coefficients and unknowns are elements in a finite field $F(p^n)$ and all but possibly c_{s+1} are different from zero; moreover, each exponent a_i is positive and less than the order of the field. The problem is to determine the number of solutions $x_i^{a_i}$ of this equation. It is shown that this problem can be reduced to the similar problem of determining the number of solutions y_1^m, y_2^m of an equation of the form $cy_1^m+dy_2^m+1=0$ with c_1, c_2, y_1, y_2 in $F(p^n)$ and $y_1y_2\neq 0$. For the simpler equation just mentioned the problem has been solved in special cases. *H. W. Brinkmann.*

Schwarz, Štefan. Contribution à la réductibilité des polynômes dans la théorie des congruences. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1939**, 7 pp. (1939). [MF 16128]

Let p be a prime, $f(x)$ a polynomial of degree $n<p$, and

$$x^{kp}\equiv\sum_{j=0}^{n-1}c_{kj}x^j \pmod{p, f(x)},\quad k=0, 1, \cdots, n-1.$$

Nº 7

If C designates the matrix of coefficients of these equations and if $f(x)=\prod_{i=1}^{m} f_i^{r_i}$, where each f_i is irreducible of degree q_i, then $|C-\lambda I|$, the characteristic polynomial of the substitution, has the form $(-1)^n\lambda^\alpha\prod_{i=1}^{m}(\lambda^{q_i}-1)$, where $\alpha=\sum_{i=1}^{m} q_i(r_i-1)$. This generalizes a result of K. Petr [Časopsis Pěst. Mat. Fys. **66**, 85–94 (1936)].
I. Niven (West Lafayette, Ind.).

Chowla, S. On difference-sets. J. Indian Math. Soc. (N.S.) **9**, 28–31 (1945). [MF 15923]

The positive integers $d_1, \cdots, d_t$, incongruent mod m, are said to form a difference set (mod m; $m_1, \cdots, m_r$), where $m\equiv 0$ (m_j), $j=1, \cdots, r$, if the number of solutions of $d_i-d_j\equiv g$ (m) is the same for all $g\not\equiv 0$ (m_j), $j=1, \cdots, r$. The author states without proof various theorems and conjectures on such difference sets. *H. B. Mann.*

Bose, R. C., and Chowla, S. On the construction of affine difference sets. Bull. Calcutta Math. Soc. **37**, 107–112 (1945). [MF 15685]

An affine difference set is a set of s integers $d_1, \cdots, d_s$ such that among the $s(s-1)$ differences d_i-d_j, $i, j=1, \cdots, s$; $i\neq j$, all the residues mod (s^2-1) which are not divisible by $s+1$ occur exactly once. Such an affine difference set always exists if s is a power of a prime. The authors describe an efficient method for obtaining an affine difference set whenever s is a prime. *H. B. Mann* (Columbus, Ohio).

Ikehara, Shikao. On Kalmár's problem in "Factorisatio Numerorum." II. Proc. Phys.-Math. Soc. Japan (3) **23**, 767–774 (1941). [MF 15006]

[Part I appeared in the same Proc. **21**, 208–219 (1939).] The following result is proved. Let $k(n)$ denote the number of representations of an integer n as a product of factors; let $K(x)=\sum_{m=1}^{[x]}k(m)$; let ρ be the positive root of the equation $\zeta(s)=2$. Then $n^{-\rho}K(n)=\rho^{-1}R^{-1}+F(n)$, where $R=-\zeta'(\rho)$, $F(n)=O\{\exp(-\alpha(\log\log n)^\gamma)\}$, $\gamma^{-1}=\frac{2}{3}+\epsilon$ for any $\epsilon>0$, and $0<\alpha<\frac{1}{2}$. The bound for $F(n)$ is an improvement on previous results. The proof uses the results of Vinogradov on Weyl's sums. *R. D. James* (Vancouver, B. C.).

Romanoff, N. P. On a special orthonormal system and its connection with theory of primes. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 353–364 (1945). (Russian. English summary). [MF 14588]

Let $B_k(z)$ be the kth Bernoullian polynomial (for example, $B_1(z)=z-\frac{1}{2}$) and let $B_k=B_k(0)$. For each k the author considers the system of functions $\theta_{n,k}(x)$ $(n=1, 2, \cdots)$ defined by $\theta_{n,k}(x)=B_k(nx-[nx])$ in the interval $0\leqq x\leqq 1$ and obtains from them a normal orthogonal set S_k of functions

$$\psi_{n,k}(x)=\left\{\frac{k!^2}{(2k)!}|B_{2k}|\varphi_{2k}(n)\right\}^{-\frac{1}{2}}\sum_{d|n}\mu(n/d)d^k\theta_{d,k}(x)$$

of $L^2(0, 1)$. Here $\mu(n)$ is the Möbius function and

$$\varphi_{2k}(n)=\sum_{d|n}\mu(n/d)d^{2k}.$$

[The right hand sides of formulae (22) and (23) on page 361 should be multiplied by $\sqrt{2}$ and by $1/\sqrt{2}$, respectively.]

The system S_k is not, as it stands, complete. However, it is shown (in detail for the case $k=1$) that the modified system consisting of unity, S_k and S_{k+1} forms a complete orthonormal system for $L^2(0, 1)$. Parseval's theorem then leads to a number of identities which may have applications to the study of the order of $\sum_{n\leq x}\mu(n)$ and to other similar problems of Diophantine approximation of the type considered by Hecke, Behnke, Hardy and Littlewood. [Cf., for example, Koksma, Diophantische Approximationen, Ergebnisse der Math., vol. 4, no. 4, Springer, Berlin, 1936, chap. 9.] *R. A. Rankin* (Cambridge, England).

★**Wintner, Aurel. Eratosthenian Averages.** Baltimore, Md., 1943. v+81 pp. $2.25.

This monograph is a systematic treatment and extension of related topics in number theory discussed in several papers by the author during the past few years. He derives the title from the fact that a restatement of the proof of the sieve of Eratosthenes is equivalent to the construction of an infinite matrix (ϵ_{nm}), $n, m=1, 2, \cdots$, where the mth column is a periodic sequence of period m, in which $m-1$ zeros are followed by a 1. The linear transformation of a sequence $f'(n)$ into $f(n)$ by this matrix, (1) $f(n)=\sum_{m=1}^{\infty}\epsilon_{nm}f'(m)$, is equivalent to (2) $f(n)=\sum_{d|n}f'(d)$, summed over all divisors d of n. The relations (2) can be inverted by the Sylvester identity $f'(n)=\sum_{d|n}\mu(d)f(n/d)$, where $\mu(n)$ is the Möbius function.

The equations (2) are equivalent to the identities (3) $\sum_{m=1}^{n}f(m)=\sum_{m=1}^{n}[n/m]f'(m)$, where $[x]$ denotes the greatest integer not exceeding x. The formulae (3) lead formally to (4) $M(f)=\sum_{n=1}^{\infty}f'(n)/n$, if $M(f)$, the mean of $f(n)$, is defined as the limit of $n^{-1}\sum_{m=1}^{n}f(m)$ as $n\to\infty$. The relation (3) can be written as a linear transformation of the partial sums of the series $\sum_{n=1}^{\infty}f'(n)/n$ into the sequence of numbers $n^{-1}\sum_{m=1}^{n}f(m)$, $n=1, 2, \cdots$, or vice versa. The author calls these two linear transformations "Eratosthenian summation methods." It is shown that these summation methods are consistent, that is, (4) holds if $M(f)$ exists and $\sum_{n=1}^{\infty}f'(n)/n$ converges, but that neither is regular in the sense of Toeplitz, that is, the convergence of $\sum f'(n)/n$ does not imply the existence of $M(f)$, or conversely. Part I of the monograph is concerned chiefly with Tauberian conditions for the validity of (4). The most difficult of these results depend on the formal relation

$$\sum f(n)r^n=\sum f'(n)r^n/(1-r^n)$$

and the theorem of Hardy and Littlewood (which is deeper than the prime number theorem) to the effect that Lambert summability implies Abel summability [Proc. London Math. Soc. (2) 19, 21–29 (1920); 41, 257–270 (1936)]. From this it is deduced, for example, that, if $f'(n)=O_L(1)$, then the existence of $M(f)$ implies the convergence of $\sum f'(n)/n$. Also, if $f(n)=O_L(1)$, then the convergence of $\sum f'(n)/n$ implies the existence of $M(f)$.

Part I also contains a discussion of the validity of Euler's formal factorization for certain Dirichlet series. If $g(n)$ is multiplicative, that is, $g(mn)=g(m)g(n)$ whenever m and n are relatively prime, then formally

$$\sum_n g(n)/n^s=\prod_p\left\{1+\sum_{k=1}^{\infty}g(p^k)/p^{ks}\right\},$$

where the product runs over all primes p. The author shows with a set of counter-examples that the convergence of either side of this formal relation does not imply the convergence of the other; nor does the convergence of both sides imply the equality. Furthermore, the infinite product need not have an abscissa of convergence, that is, it may converge at $s=1$ without converging for all $s>1$. [Cf. Wintner, Duke Math. J. 11, 277–285 (1944); these Rev. 5, 255.] A criterion for the validity of this formal relationship given in theorem V, page 17 is incorrect.

Part II is a study of the almost periodic properties of $f(n)$ generated by the Eratosthenian linear transformation (1). Let $c_m(n)$ denote the Ramanujan sum $\sum\exp(2\pi iln/m)$ summed over those l satisfying $1\leq l\leq m$ and $(l, m)=1$. It is shown that, if $\sum f'(n)/n$ is absolutely convergent, then $f(n)$ is almost periodic (B) and has the Fourier expansion $f(n)\sim\sum_{m=1}^{\infty}a_mc_m(n)$, where the Fourier coefficients $a_m=\sum_{n=1}^{\infty}f'(mn)/mn$, so that $a_1=M(f)=\sum f'(n)/n$. Examples show that this Fourier series may diverge for all n; however, if $d(n)$ is the number of divisors of n and if $\sum d(n)f'(n)/n$ converges absolutely, then the Fourier series converges absolutely to $f(n)$ for all n. Otherwise the situation is quite complicated. For example, there exist series $\sum a_mc_m(n)$ convergent for all n to functions which are not almost periodic (B) or do not even have a mean $M(f)$; or, an almost periodic (B) function $f(n)$ may possess a Fourier series $\sum a_mc_m(n)$ which converges for all n, but whose sum is not $f(n)$; or, an almost periodic (B) function $f(n)$ need not have a Fourier series of the form $\sum a_mc_m(n)$; etc. A sufficient condition for $f(n)$ to be uniformly almost periodic is that $\sum|f'(n)|$ is convergent. The above results are illustrated by deriving the Ramanujan identities for $\sigma_a(n)$ and for the Jordan generalization of the Euler ϕ-function, $\phi_a(n)$; the Cantor identity for the function $\gamma(n)$, which may be interpreted as the number of Abelian groups of order n; and the almost periodic nature of the Gram function defined by $F(n)=n-F([n/2])$.

The beginning of part III is a discussion from the above point of view of Chebyshev's function $\Lambda(n)$. Furthermore, let $M(\Lambda^{(l)})$ denote the auto-correlation of Λ, that is, the mean of $\Lambda(n)\Lambda(n+l)$ for a fixed $l>0$. This function is shown to be closely connected with Sylvester's unproved assertion on Goldbach's problem, namely, $\rho(2n)\sim Cf(n)n/\log^2 n$, where $\rho(n)$ is the number of representations of n as the sum of two primes and $f(n)=\prod_{2<p|2n}(p-1)(p-2)^{-1}$. In fact, $M(\Lambda^{(2n)})=Cf(n)$ and $\rho(2n-1)=M(\Lambda^{(2n-1)})=0$. The function $f(n)$ is almost periodic (B) with a Fourier series of the form $\sum a_mc_m(n)$, which is absolutely convergent to $f(n)$.

The second section of part III deals with a discussion of the relationship between Poisson's statistical law of rare events and the distribution of arbitrary positive additive functions satisfying $f(p)=1$ for all primes p. [Cf. Wintner, Duke Math. J. 9, 425–430 (1942); these Rev. 3, 271.]

The third section is devoted to the statistics of the sum of two squares. Let $G_l(n)$ denote the number of integers not exceeding n which have exactly l representations as the sum of two squares. It is shown that for every $l>0$ either $G_l(n)\sim C_l(n/\log n)^{\frac{1}{2}}$, $G_l(n)\sim C_ln/\log n$ or

$$G_l(n)\sim C_l(n/\log n)^{\frac{1}{2}}\log\log n,$$

depending on the arithmetical structure of l. This theorem depends on a generalization of the Ikehara-Wiener Tauberian theorem [Wintner, Amer. J. Math. 64, 320–326 (1942); these Rev. 3, 271].

The last section deals with $\mu_R(n)$, the Möbius function corresponding to an arbitrary set R of primes, that is, $\mu_R(n)$ is zero if n is not square-free or has a prime factor not in R and otherwise $\mu_R(n)=(-1)^\nu$ if ν denotes the number of prime factors of n. It is proved that, for every set R, $\sum\mu_R(n)/n$ is convergent; its sum is zero if and only if $\sum 1/r=\infty$, where the sum is taken over all primes r in R.

This, of course, is precisely the prime number theorem if R is the set of all primes. The result is not proved by the use of Ikehara's theorem, but via a weakened form of the Hardy-Littlewood theorem concerning Lambert summability mentioned above [cf. the following review]. An appendix contains a proof of this Hardy-Littlewood theorem.

This monograph is interestingly written from the point of view of historical references and by reason of the fact that each theorem is discussed with the aid of numerous counter-examples to show the limitations or possibilities of improvement. Each result is well illustrated by applications to the number-theoretic functions; short and simplified proofs are obtained for many standard theorems.

P. Hartman (Flushing, N. Y.).

★**Wintner, Aurel. The Theory of Measure in Arithmetical Semi-Groups.** Baltimore, Md., 1944, v+56 pp. $2.25.

The object of this monograph is the study of the properties of the distribution of the primes when the sequence of positive integers is replaced by an arbitrary arithmetical semi-group. Although this monograph is self-contained, it is best read after the one reviewed above. Chapter I is a discussion of the Eratosthenian summation methods defined in the preceding review. The main Tauberian results are proved again and restated in a form convenient for the sequel.

Chapter II deals with the "zeta functions" $\zeta_R(s)$ belonging to an arbitrary set of primes R. The function $\zeta_R(s)$ and the Möbius sequence $\mu_R(n)$ may be considered to be defined by $1/\zeta_R(s)=\sum\mu_R(n)/n^s=\prod(1-r^{-s})$, where the product is taken over all primes r in the set R. A corresponding "prime number theorem," namely, that the series for $1/\zeta_R(s)$ is convergent at $s=1$, is proved for every set R. This is deduced not from Ikehara's theorem but from the "equivalent Tauberian" theorem of Wiener to the effect that, if a series has half-bounded terms, then Lambert summability implies Abel summability. [Cf. Wiener, The Fourier Integral and Certain of its Applications, Cambridge University Press, 1933, pp. 116–120.] This is a weakened form of the deeper Abelian theorem of Hardy and Littlewood which states that the "half-boundedness" condition is not needed. The superiority of this approach is illustrated by the fact that an application of Ikehara's theorem would involve the difficult task of considering the behavior of $1/\zeta_R(s)$ near the line $s=1$, while examples are given to show that the series for $1/\zeta_R(s)$ need not converge at all points of this line, and, indeed, $\zeta_R(s)$ may have this line as its natural boundary. Although all "prime number theorems" are true in terms of the Möbius function $\mu_R(n)$, the same statement cannot be made about the corresponding statements of the prime number theorem in terms of, say, the "Chebyshev function" $\Lambda_R(n)$.

The author points out that, while the convergence of "almost all" of the Dirichlet series $\sum\pm n^{-s}$ in the half-plane $\sigma>\frac{1}{2}$ has been interpreted by many authors as the "truth of Riemann's hypothesis for almost all zeta functions," this formulation ignores the arithmetical side of the problem. He suggests introducing the arbitrary $\pm$ signs in another way; namely, by considering the set of zeta functions

$$1/\zeta_R(s)=\prod\{1-(\tfrac{1}{2}\pm\tfrac{1}{2})p^{-s}\},$$

where a particular sequence of $+$ and $-$ signs determines a set R of primes r, and so $1/\zeta_R(s)=\prod(1-r^{-s})=\sum\mu_R(n)/n^s$. Also, the coefficients $R^*(n)$ in the corresponding Dirichlet series $\zeta_R(n)=\sum R^*(n)/n^s$ now form a multiplicative sequence. The abscissa of convergence for almost all $\zeta_R(s)$ and $1/\zeta_R(s)$ is $\sigma=1$. But it is proved that, for almost all sets R, $\zeta_R(s)/\zeta(s)^{\frac{1}{2}}$ is regular and distinct from zero in the half-plane $\sigma>\frac{1}{2}$, so that the corresponding Riemann hypothesis is true if and only if it is true for $\zeta(s)=\sum n^{-s}$ itself. It is furthermore shown that almost all $\zeta_R(s)$ are meromorphic in the half-plane $\sigma>\frac{1}{2}$ and possess this line as their natural boundary.

Chapter III is devoted to the study of the asymptotic behavior of the functions $\pi_R(x)$ and $[x]_R$ for arbitrary sets R of primes, where $\pi_R(x)$ is the number of primes r in R which do not exceed x and $[x]_R$ is the number of integers which do not exceed x and whose only prime divisors are in R. It is shown that, if λ_R is the convergence exponent of the set R of primes and if $\lambda_R>0$, then

$$\log\int_1^x u^{-\lambda_R}d[u]_R\sim\int_1^x -\log(1-u^{-\lambda_R})d\pi_R(u),\qquad x\to\infty;$$

that if $\zeta_R(\lambda_R)=\infty$, then

$$\log\int_1^x u^{-\lambda_R}d[u]_R\sim\int_1^x u^{-\lambda_R}d\pi_R(u),\qquad x\to\infty;$$

while if $\lambda_R=0$ and R is not a finite set, then

$$\pi_R(x)\sim\omega_R(1/\log x),\quad [x]_R\sim\zeta_R(1/\log x),$$

where $\omega_R(s)=\int_1^\infty u^{-s}d\pi_R(u)$. The methods of this chapter are employed in a discussion of the integers n for which the regular $n$gon is constructible, in the terms of the set of primes in the sequence obtained by adding 1 to a 2^mth power of 2. It is also shown that, for almost all sets R of primes, one has, in addition to the elementary result $\pi_R(x)\sim x\log x/2$, the relation $[x]_R\sim x/\Gamma(\frac{1}{2})\log^{\frac{1}{2}}x$.

The remarks made at the end of the preceding review, concerning historical references, counter-examples and illustrative applications, are applicable to this monograph also. However, this monograph is marred by an error in the formula line at the top of page 45. This makes the proof of the theorem (I) on this page incorrect. Actually, that (I) and the theorems of chapter IV which depend on it are false may be deduced from the asymptotic formulae for almost all $[x]_R$ and $\pi_R(x)$ mentioned above. There are also several trivial misstatements; these are the characterization of completely multiplicative sequences on the top of page 18; the definition of semi-groups in the first paragraph of § 17, page 18; and the replacement of $d\{u\}_Q$ by $d[u]_Q$ in formula (77).

P. Hartman (Flushing, N. Y.).

★**Wintner, Aurel. An Arithmetical Approach to Ordinary Fourier Series.** Baltimore, Md., 1945. 29 pp. $1.20.

The intention of the author is to present "in a form acceptable to a mathematical conscience" some techniques and formulae which have been used for a long time by astronomers for problems such as the numerical computation of Fourier coefficients. Actually, he goes somewhat beyond this in the last part of the paper, where he uses Tauberian theorems developed in the first part of each of the two monographs reviewed above.

Let $f(x)$ be a periodic function of period 1. Let $f_n(x)=\sum_{m=1}^n f(x+m/n)/n$, so that, if $f(x)$ is Riemann integrable, $f_n(x)\to\int_0^1 f(t)dt$ as $n\to\infty$ for all x. It is first shown that, if $f(x)$ has a derivative satisfying a uniform Lipschitz condition and if $a_0=\int_0^1 f(t)dt=0$, then, if $\mu(n)$ denotes the Möbius function, each of the series $g_n(x)=\sum_{k=1}^\infty\mu(k)f_{nk}(x)$ is absolutely and uniformly convergent and one has the absolutely and uniformly convergent expansion $f(x)=\sum_{n=1}^\infty g_n(x)$; furthermore, $g_n(x)$ must be the function $a_n\cos 2\pi nx+b_n\sin 2\pi nx$

when $f(x)$ is a trigonometric polynomial. The proof depends merely on the estimate $d(k)=O(k^\epsilon)$, where $d(k)$ is the sum of the divisors of k, and a rearrangement of an absolutely convergent double series. It follows that $g_n(x)=a_n \cos 2\pi nx + b_n \sin 2\pi nx$ is also true for all functions $f(x)$ of the type considered in the first part of the theorem if one uses the fact that this type of function can be approximated strongly by trigonometric polynomials.

But in this direction one can go even further. The conditions on the smoothness of $f(x)$ can be replaced by the assumption that $f(x)$ has a Fourier series

$$\sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

such that

$$\sum_{n=1}^{\infty} 2^{\nu(n)}(|a_n|+|b_n|)<\infty;$$

here $\nu(n)$ is the number of distinct prime divisors of n. Also, since $2^{\nu(n)}=O(n^\epsilon)$, $\sum_{n=1}^{\infty} n^\epsilon(|a_n|+|b_n|)<\infty$ is sufficient; this condition is fulfilled if $f(x)$ satisfies the uniform fractional Lipschitz condition $|f(x_1)-f(x_2)|<C|x_1-x_2|^{\frac{1}{2}+\epsilon}$. It is also shown that the condition $\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty$ stated by Bruns [Grundlinien des Wissenschaftlichten Rechnens, Leipzig, 1903] is not sufficient. The series for $g_n(x)$ at $x=0$ gives the Bruns formula $na_n=\sum_{k=1}^{\infty}(\mu(k)/k)\sum_{m=1}^{kn} f(m/kn)$ for the Fourier coefficient a_n (no value of x gives a similar formula for all b_n).

Using known relations between the order of magnitude of Fourier coefficients and smoothness properties of $f(x)$, one can use the above results to obtain conditions for the smoothness of $f(x)$ in terms of the error $f_n(x)-\int_0^1 f(t)dt$. For example, $f(x)$ is analytic along the x-axis if and only if this error is uniformly $o(\theta^n)$ for some θ, $0<\theta<1$, as $n\to\infty$.

The derived formula implies a generalization of the functional equation $B_j(n)=n^{j-1}\sum_{m=0}^{n-1}B_j(x+m/n)$ of the Bernoulli polynomials $B_1(x), B_2(x), \cdots$. Let $\alpha, \beta, \lambda_1, \lambda_2, \cdots$ be constants such that the series $f(x)=\sum(\alpha\lambda_n \cos 2\pi nx+\beta\lambda_n \sin 2\pi nx)$ is convergent almost everywhere and let the relation $\lambda_{jk}=\lambda_j\lambda_k$ hold for all positive integers j, k; then the functional equation $f_n(x)=\lambda_n f(nx)$ holds for almost all x and for any integer $n>0$.

A simple answer to a problem in interpolation is given as follows. If $f(x)$ has an absolutely convergent Fourier series and if $f^n(x)$ is the trigonometrical polynomial of degree n such that $f^n(x)=f(x)$ at the equidistant points $x=m/(2n+1)$, $m=0, 1, \cdots, 2n$, then $f^n(x)\to f(x)$ uniformly for all x as $n\to\infty$.

The proofs up to this point are "elementary." Using the Kronecker-Weyl theorem and a Tauberian theorem he derived [p. 20 in the first of his monographs reviewed above, p. 6 in the second], the author shows that, if $f(x)$ is any Riemann integrable function of period 1, then $\sum_{n=1}^{\infty} n^{-1}\sum_{d|n}\mu(n/d)f(xd)$ converges for all irrational x and its sum is $\int_0^1 f(t)dt$. This result contains the prime number theorem. By replacing $f(x)$ by $f(x)\exp(-2\pi inx)$, one obtains a formula for the Fourier coefficient c_n occurring in the expansion $f(x)\sim\sum c_n \exp(2\pi inx)$.

An application of the last theorem to the first Bernoulli function $B_1(t)=t-[t]-\frac{1}{2}=-\pi^{-1}\sum \sin 2\pi mt/m$ yields the following theorem on Farey sections. If $\phi^x(n)$ denotes the number of elements of the nth Farey section which do not exceed x (so that $\phi^1=\phi$, the Euler ϕ-function), then the series $\sum\{x\phi(n)/n-\phi^x(n)/n\}$ converges for all irrational x, $0<x<1$, to $\frac{1}{2}$. *P. Hartman* (Flushing, N. Y.).

Delaunay, B. Local method in the geometry of numbers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 241–256 (1945). (Russian. English summary) [MF 15356]

In recent years, the problem of the lattices of smallest determinant satisfying certain conditions (for example, to contain no inner points of a given region except the origin) has been studied particularly by H. Davenport, L. J. Mordell, K. Mahler and B. Segre [see Davenport, J. London Math. Soc. 16, 98–101 (1941); 18, 168–176 (1943); Mordell, J. London Math. Soc. 16, 152–156 (1941); 17, 107–115 (1942); 18, 201–210, 210–217 (1943); 19, 92–99 (1944); Proc. London Math. Soc. (2) 48, 198–228 (1943); 339–390 (1945); Mahler, J. London Math. Soc. 17, 130–133 (1942); 18, 233–238 (1943); Proc. Cambridge Philos. Soc. 40, 107–116, 116–120 (1944); Duke Math. J. 12, 367–371 (1945); Segre, Duke Math. J. 12, 337–365 (1945); these Rev. 3, 70; 5, 254; 3, 167; 4, 131; 6, 37, 257; 5, 172; 6, 257; 4, 212; 6, 119, 258]. The author tries to solve problems of this kind by what he calls the "local method." He replaces the variable lattice by the configuration of only a finite number of points in the lattice and satisfies the extremal conditions by varying the configuration; if the lattice determined by the configuration is admissible, then the problem is solved. This is carried out in detail for the region $|xy|\leqq 1$ connected with Hurwitz's theorem on $|\alpha-p/q|<1/(q^2\sqrt{5})$ and for the region $|xy(x+y)|\leqq 1$ considered by Mordell. [The local method is closely connected with the method of Mordell. Moreover, there are simple regions like $x^2(x^2+y^2)\leqq 1$ for which this method cannot lead to a solution since the critical lattices have no points on the boundary of the region.] *K. Mahler* (Manchester).

Mahler, K. On lattice points in a cylinder. Quart. J. Math., Oxford Ser. 17, 16–18 (1946). [MF 15892]

If K denotes a convex body in 3-space symmetric with respect to the origin, a lattice Λ whose only point interior to K is the origin is known as a K-admissible lattice. Let $d(\Lambda)$ be the determinant of such a lattice. Let $\Delta(K)$ be the lower bound of $d(\Lambda)$ for all K-admissible lattices Λ. This paper shows that $\Delta(K)=\frac{1}{2}\sqrt{3}$ when K is the cylinder $x_1^2+x_2^2\leqq 1$, $|x_3|\leqq 1$. The proof depends on a bound for the number of nonoverlapping circular cylinders of the same dimensions which can be placed in a cube with axes perpendicular to a face of the cube. This is a consequence of an analogous result for a plane from an earlier paper of Segre and Mahler [Amer. Math. Monthly 51, 261–270 (1944); these Rev. 6, 16]. *D. Derry* (Vancouver, B. C.).

Remak, Robert. Ein Satz über die sukzessiven Minima bei definiten quadratischen Formen. Nederl. Akad. Wetensch., Proc. 44, 1071–1076 (1941). [MF 15768]

Let $\lambda_1, \cdots, \lambda_n$ be the successive minima associated with a convex body, centre at the origin, in n-dimensional space, as defined by Minkowski. For the special case when the body is given by $Q\leqq 1$, where Q is a positive definite quadratic form in n variables, of determinant D, the author proves that $\lambda_1^2\cdots\lambda_n^2\leqq\gamma_n^n D$. Here γ_n is the greatest minimum of any positive definite quadratic form of determinant 1. The result was already proved, in effect, by Minkowski [Geometrie der Zahlen, Leipzig, 1910, § 51]; the present proof differs from Minkowski's only in its notation. *H. Davenport* (London).

★**Eichler, Martin. Zur Theorie der quadratischen Formen gerader Variablenzahl.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 34–46, Füssli, Zürich, 1945.

It has long been known that there exists a very close relationship between the arithmetic theory of binary quadratic forms and the arithmetic of quadratic fields. More recently Brandt has shown [Math. Ann. **99**, 1–29 (1928); **94**, 179–197 (1925)] that a similar connection exists between the ideal theory in quaternion algebras and the quaternary quadratic forms. Hecke's studies of the number of representations of integers by quadratic forms in $2n$ variables seemed to point to the existence of a generalization of these results [cf. Hecke's Lectures on Dirichlet Series, Institute for Advanced Study, Princeton, N. J., 1938]. The paper under review contains a condensed exposition of such a generalization for quadratic forms in $2n$ variables. Following Brandt [Jber. Deutsch. Math. Verein. **47**, 149–159 (1937)] the author studies "stem forms" (Stammformen), that is, integral forms of smallest discriminant into which an integral form may be transformed by a rational transformation and cancellation of the g.c.d. of the coefficients. He introduces a canonical form into which a stem may be transformed by an integral p-adic transformation and, employing suitable arithmetic invariants, classifies stem forms into orders and genera. Next, he considers the set $f_j = x'A_jx$ $(j=1, \cdots, h)$ of representatives of classes of stem forms of a given order and defines a "transformer" [cf. Brandt, loc. cit.] of norm t as a linear transformation T_{ik} for which $T_{ik}'A_iT_{ik} = tA_k$. The concept of the transformer is the pivotal concept of the theory. Multiplication of transformers and their decomposition into primes obey the same laws as the ideals of a simple algebra [cf. M. Eichler, S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 1–28].

In the concluding two sections the author employs his theory to give a new proof for certain forms in $2n$ variables of A. Meyer's theorem [Vierteljschr. Naturforsch. Ges. Zürich **36**, 241–250 (1891)] that in general two indefinite quadratic forms of the same genus are equivalent. To the best of the reviewer's knowledge this is the first essentially new approach to the problem posed by Meyer. It is the hope of the author that the restrictive hypotheses of his present proof will be removed in the near future.

A. E. Ross (St. Louis, Mo.).

Reiner, Irving. On genera of binary quadratic forms. Bull. Amer. Math. Soc. **51**, 909–912 (1945). [MF 14459]

Employing Dirichlet's theorem on the infinity of primes in an arithmetic progression, the author proves two well-known theorems of Gauss [Disquisitiones Arithmeticae, arts. 234–265], one relating to the existence of genera of properly primitive binary quadratic forms and the other on the equality of the number of classes of such forms in different genera of the same determinant. The proof of the second theorem is made to depend upon the existence of a linear transformation of prime determinant p which carries forms β_1, β_2 of one genus into forms $p\phi_1, p\phi_2$, where ϕ_1, ϕ_2 belong to another preassigned genus and where the equivalence of ϕ_1 and ϕ_2 implies the equivalence of β_1 and β_2. *A. E. Ross.*

Rédei, L. Über den geraden Teil der Idealklassengruppen in algebraischen Zahlkörpern. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 829–841 (1940). (Hungarian. German summary) [MF 15563]

Let k be an algebraic number field of finite degree, k_n an unramified cyclic extension of degree 2^n relative to k; let $n>1$ and k_1 be given and consider all k_n containing k_1. The author obtains a necessary and sufficient condition for the existence of at least one k_n in terms of Artin symbols, generalizing the results of Reichardt [J. Reine Angew. Math. **170**, 75–82 (1933)] concerning quadratic fields k.

C. L. Siegel (Princeton, N. J.).

Papkovi, P. On imaginary quadratic fields admitting only ambiguous classes. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 585–592 (1944). (Georgian and Russian) [MF 14608]

The author states that each genus of an imaginary quadratic field $K(\sqrt{d})$ contains only one ideal class if and only if $P \geqq \frac{1}{2}(1+D)^{\frac{1}{2}}$, where P is the smallest prime in $K(\sqrt{d})$ for which the Kronecker symbol $(d|P)=+1$ and $D=|d|=p_1 \cdots p_t \equiv 3 \pmod 4$. Next, generalizing the well-known results of Frobenius [S.-B. Preuss. Akad. Wiss. **1912**, 966–980; cf. also G. Rabinovitsch, J. Reine Angew. Math. **142**, 153–164 (1913)], he states that, in order that there should be a single ideal class in each genus, it is necessary and sufficient that for integral values of x in the interval $(1, R-1)$ the trinomial $Q(x^2-x)+R$ should take on values each of which is either a prime or a prime multiplied by a divisor of the discriminant. Here $Q|D$, $Q<D^{\frac{1}{2}}$, $R=\frac{1}{4}(Q+Q')$ is a prime, $QQ'=D$. The author further asserts that the interval $(1, R-1)$ in this theorem may be replaced by the interval $(1, \Delta)$, where $\Delta=(2Q)^{-1}\{\frac{1}{3}(4Q-3)D\}^{\frac{1}{2}}+1$. A list of examples of imaginary quadratic fields with but a single class in each genus is given at the end of § 1 [cf. tables in Sommer's Vorlesungen über Zahlentheorie, Teubner, Leipzig, 1907, pp. 346–354, and in Dickson's Introduction to the Theory of Numbers, University of Chicago Press, 1929, pp. 85–89]. *A. E. Ross* (St. Louis, Mo.).

Jarník, Voitech. Sur les approximations diophantiques des nombres p-adiques. Revista Ci., Lima **47**, 489–505 (1945). [MF 15432]

Let s be a positive integer and $\lambda(n)$ a positive function of the integer $n \geqq 1$ such that $n^{s+1}\lambda^s(n)$ decreases steadily to zero as n tends to infinity. Denote by $H(\lambda)$ and $h(\lambda)$ the sets of all systems of s p-adic integers $\{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_s\}$ for which the conditions

$$(p, n)=1; \quad |n\mathfrak{a}_i - y_i|_p < \lambda(\max(|n|, |y_1|, \cdots, |y_s|)), \qquad i=1, \cdots, s,$$

have an infinite or a finite number of solutions in rational integers $n, y_1, \cdots, y_s$, respectively. Define the external measures $\mu H(\lambda)$ and $\mu h(\lambda)$ in analogy to the real case; then $\mu H(\lambda)=0$ if $\sum_1^\infty n^s\lambda^s(n)$ converges, $\mu h(\lambda)=0$ if $\sum_1^\infty n^s\lambda^s(n)$ diverges. This result corresponds to a well-known theorem of Khintchine for the real case [Math. Z. **24**, 706–714 (1926); see also V. Jarník, Math. Z. **33**, 505–543 (1931); A. Grošev, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1937**, 427–443 (1937)]. For the theory of measure in the p-adic field, see also the Amsterdam thesis of H. Turkstra [Metrische Bijdragen tot de Theorie der Diophantische Approximaties in het Lichaam der P-adische Getallen, Groningen, 1936]. *K. Mahler* (Manchester).

Koksma, J. F. On decimals. Nieuw Arch. Wiskunde (2) **21**, 242–267 (1943). [MF 15706]

Let θ be a real number, a an integer not less than 2 and let, as usual, $(\theta a^n) = \theta a^n - [\theta a^n]$ denote the fractional part of

$\theta\alpha^n$. Let $\nu_{\alpha,\beta}(N)$ be the number of numbers $(\theta\alpha^n)$, $1\leqq n\leqq N$, which belong to the interval (α,β) interior to the interval $(0,1)$. Let $\nu_{\alpha,\beta}(N)=(\beta-\alpha)N+R_{\alpha,\beta}(N)$. By a classical theorem of Weyl, $R_{\alpha,\beta}(N)=o(N)$ for almost all θ. The present paper gives a number of results on the order of magnitude of $R_{\alpha,\beta}(N)$ as well as on the order of magnitude of $R^*(N)=\sum_{n=1}^N((\theta\alpha^n)-\frac{1}{2})$ and $Q(N)=\sum_{n=1}^N(\rho_n-n/N)^2$, where $\rho_1,\cdots,\rho_N$ denote the first N numbers $(\theta\alpha^n)$ arranged in nondecreasing order of magnitude. We quote the following results; there are many more general theorems which there is not space to state here. For almost all θ,

$$R^*(N)=o(N^{\frac{1}{2}}(\varphi(N))^{\frac{1}{2}}),$$
$$Q(N)=o(N^{\frac{1}{2}}(\varphi(N))^{\frac{1}{2}}),$$
$$R_{\alpha,\beta}(N)=o(N^{\frac{1}{2}}(\varphi(N))^{\frac{1}{2}}),$$

where $\varphi(N)$ is any positive, nondecreasing function of n such that $\sum 1/(n\varphi(n))<\infty$ and $\varphi(n+1)\leqq(1+K/n)\varphi(n)$, K being independent of N. *R. Salem.*

Koksma, J. F. A general theorem from the theory of uniform distribution modulo 1. Mathematica, Zutphen. B. 11, 7–11 (1942). (Dutch) [MF 15731]

Koksma, J. F. Some integrals in the theory of uniform distribution modulo 1. Mathematica, Zutphen. B. 11, 49–52 (1942). (Dutch) [MF 15729]

If $f(k)$ is a real function of the integral variable k, let just $N_{\alpha,\beta}(N)$ of the numbers (1) $f(k)-[f(k)]$, $k=1,\cdots,N$, lie in the interval $\alpha\leqq u<\beta$. Put $R_{\alpha,\beta}(N)=N_{\alpha,\beta}(N)-(\beta-\alpha)N$ and denote by $D(N)$ the upper bound of $R_{\alpha,\beta}(N)/N$ for $0\leqq\alpha\leqq\beta\leqq1$. In another paper the author will show that, if $w(t)$ is of period 1 and of total variation T in $0\leqq t\leqq1$, then

$$\left|\sum_{k=1}^N w(f(k))-N\int_0^1 w(t)dt\right|\leqq TND(N).$$

From this result he deduces, in the first note, that

$$\int_0^1 R_{0,t}(N)dw(t)=N\int_0^1 w(t)dt-\sum_{k=1}^N w(f(k)).$$

In the second note he obtains some further identities, for example,

$$\int_0^1 R_{0,t}^2(N)dt=\tfrac{1}{3}N^2+N\sum_{k=1}^N f^2(k)$$
$$+\sum_{k=1}^N f(k)-2\sum_{k=1}^n\sum_{h=1}^k \max(f(h),f(k)).$$

K. Mahler (Manchester).

Koksma, J. F. Sur la théorie métrique des approximations diophantiques. Nederl. Akad. Wetensch., Proc. 48, 249–265 = Indagationes Math. 7, 54–70 (1945). [MF 15797]

The paper gives a new extension of the well-known theorem of Khintchine on the metric theory of Diophantine approximation. In particular, the following theorem is proved. Let $f(n)$ be a positive function of the integer n such that $f(n+1)-f(n)$ is larger than a positive constant independent of n. Let $g(n)$ be a positive function of n such that $f(n)g(n)$ is monotonic nondecreasing for large n. Let α be a given real number and let $\varphi(n)$ be positive and monotonic nonincreasing. Then for almost all $\theta\geqq1$ the inequality

$$-\varphi(n)\leqq g(n)\theta^{f(n)}-p-\alpha\leqq\varphi(n)$$

has only a finite number of integral solutions in p, n $(n\geqq1)$ if $\sum\varphi(n)<\infty$, whereas both inequalities

$$0<g(n)\theta^{f(n)}-p-\alpha<\varphi(n),$$
$$-\varphi(n)<g(n)\theta^{f(n)}-p-\alpha<0$$

have an infinite number of solutions if $\sum\varphi(n)=\infty$.

Another result is the following: α and $\varphi(n)$ having the same meaning and ω being not greater than 1, if $\sum\varphi([n^{1/\omega}])=\infty$, then for almost all $\theta\geqq1$ both inequalities

$$\alpha<\theta^{n^\omega}-p<\alpha+\varphi(n),\ \alpha-\varphi(n)<\theta^{n^\omega}-p<\alpha$$

have an infinite number of solutions in p, n $(n\geqq1)$. The paper gives other more general theorems of the same type, which there is not space to state here. *R. Salem.*

Wade, L. I. Remarks on the Carlitz ψ-functions. Duke Math. J. 13, 71–78 (1946). [MF 15876]

Let $GF(p^n)$, $GF[p^n,x]$ and $GF(p^n,x)$ be, respectively, the finite field of p^n elements, p prime, the ring of polynomials in the indeterminate x over $GF(p^n)$ and the field of quotients of $GF[p^n,x]$. The author proves theorems concerning power series in a variable t over an algebraically closed field $\mathfrak{M}$ with a non-Archimedian valuation $v(a)$, $a\epsilon\mathfrak{M}$, with respect to whose associated topology, $m(a,b)=v(a-b)$, $\mathfrak{M}$ is complete. In particular, let $\mathfrak{M}$ be the essentially unique algebraically closed complete extension of $GF(p^n,x)$ with its valuation $v(E/G)=p^{e-g}$, where e and g are the degrees of E and G of $GF[p^n,x]$. The author proves that the function defined by

$$(1)\qquad \psi(t)=\sum_{j=0}^\infty(-1)^j t^{p^{nj}}/F_j,$$
$$F_0=1,\quad F_j=[j][j-1]^{p^n}\cdots[1]^{p^{n(j-1)}},$$
$$j\geqq1,\quad [j]=x^{p^{nj}}-x,$$

converges for all $t\epsilon\mathfrak{M}$ and is expressible as the product

$$(2)\qquad \psi(t)=t\prod(1-t/E\xi)$$

taken over all E of $GF[p^n,x]$, where ξ in $\mathfrak{M}$ is any root of

$$\xi^{p-1}=F_1\textstyle\sum'1/\xi^{p^n-1},$$

the summation extending over all primary (monic) E of $GF[p^n,x]$. Carlitz [same J. 1, 137–168 (1935)] derived (1) from his definition (2) and discussed the infinitely many valued inverse function $\lambda(t)$, where $\lambda(\psi(t))=t$ for all t. For $\lambda(t)$, the author proves that, in the neighborhood of every $\alpha\epsilon\mathfrak{M}$,

$$\lambda(t)=\lambda(\alpha)+\sum_{j=0}^\infty(t-\alpha)^{p^{nj}}/L_j,$$
$$L_0=1,\quad L_j=[j][j-1]\cdots[1],\qquad j\geqq1.$$

R. Hull (Lincoln, Neb.).

Wade, L. I. Transcendence properties of the Carlitz ψ-functions. Duke Math. J. 13, 79–85 (1946). [MF 15877]

The author applies the "Siegel-Schneider method" [Schneider, J. Reine Angew. Math. 172, 65–69, 70–74 (1934)], by way of several lemmas, to prove the following theorem for the function $\psi(t)$ [see the preceding review]. If α and β are elements of $\mathfrak{M}$, $\alpha\neq0$, or $\lambda(0)\neq0$ if $\alpha=0$, $\beta\epsilon GF(p^n,x)$, then at least one of α, β, $\psi(\beta\lambda(\alpha))$ is transcendental over $GF(p^n,x)$, where $\lambda(\alpha)$ is any fixed determination of the infinitely many valued function $\lambda(\alpha)$. *R. Hull.*

THEORY OF GROUPS

Miller, G. A. Prime number of conjugate operators in a group. Proc. Nat. Acad. Sci. U. S. A. **32**, 53–56 (1946). [MF 15517]

If every noninvariant operator of a finite group G has exactly p conjugates, then G is the direct product of its Sylow subgroup of order a power of p and an Abelian subgroup whose order is prime to p. The central quotient group of G is Abelian of order p^m and type 1^m, where m is even.

G. de B. Robinson (Toronto, Ont.).

Zhdanov, G. S. The numeral symbol of close packing of spheres and its application in the theory of close packings. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 39–42 (1945). [MF 15228]

Belov has shown that the possible close packings of spheres belong to the space groups $C6/mmc$, $C6mc$, $C\bar{3}m$, $C\bar{6}m$, $C3m$, $R\bar{3}m$, $R3m$ and the cubic $Fm3m$. The author observes that a close packing of spheres is uniquely determined by the thickness value of the successive layers with parallel and antiparallel packing $\cdots, p_1, p_2, p_3, \cdots, p_j, \cdots, p_n$, where p_j is equal to the number of elementary layers of spheres in a layer j with identical types of packing. Packings with a periodically recurring succession of layer thicknesses and recurring layer orientation are characterised by the shortest set of numbers which recurs infinitely in the structure. The minimum periodic set of numbers with an even number of terms in the period forms the numeral symbol of the packing. The symmetry of the numeral symbol determines the symmetry elements of the packing and thus enables it to be referred to its appropriate space group. Each of the noncubic Belov space groups contains an infinite number of close packings. The numeral symbols of these and the ratio of the number of "hexagonal" to "cubic" spheres in each are tabulated up to and including 8-layer periodicity. The total number of packings up to this stage is 34. *S. Melmore.*

★Burckhardt, Johann Jakob. Die Bewegungsgruppen der doppelt zählenden Ebene. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 153–159, Füssli, Zürich, 1945.

Here the 80 of the 230 crystallographic space groups whose lattices lie in two parallel planes are deduced by the systematic application of two procedures: (1) forming the direct product of one of the plane groups G with the group generated by S^0, S^1, S^2, S^d, where $S^0:(x, y, z)\rightarrow(x, y, \bar{z})$: $S^1:(x, y, z)\rightarrow(x+\frac{1}{2}, y, \bar{z})$; $S^2:(x, y, z)\rightarrow(x, y+\frac{1}{2}, \bar{z})$; $S^d:(x, y, z)\rightarrow(x+\frac{1}{2}, y+\frac{1}{2}, \bar{z})$; (2) if U is a subgroup of G of index two and $G=U+N$, constructing the group $U+(x, y, \bar{z})N$.

G. de B. Robinson (Toronto, Ont.).

Brauer, Richard, and Tuan, Hsio-Fu. On simple groups of finite order. I. Bull. Amer. Math. Soc. **51**, 756–766 (1945). [MF 13616]

The introduction to the paper contains the following summary of its contents. "If G is a non-cyclic simple group of order $g=pq^bg^*$, where p and q are two primes and where b and g are positive integers with $g^*<p-1$, then $G\cong LF(2, p)$ with $p=2^m\pm1$, $p>3$, or $G\cong LF(2, 2^m)$ with $p=2^m+1$, $p>3$; conversely, these groups satisfy the assumptions. As an application, we determine all simple groups of order prq^b, where p, r, q are primes and where b is a positive integer. The only simple groups of this type are the well known groups of order 60 and 168." *G. de B. Robinson.*

Zappa, Guido. Sulla risolubilità di taluni gruppi finiti. II. Boll. Un. Mat. Ital. (2) **4**, 16–25 (1942). [MF 16051]

[Cf. the same Boll. (2) **3**, 19–27 (1940); these Rev. **3**, 193.] Proof of the solubility of every group whose order $\prod_{i=1}^{r} p_i^{a_i}$, p_i primes, $p_i<p_{i+1}$, meets at least one of the following requirements: (a) $a_i\leqq3$ for $i<r$, $r<p_1$, $a_r\leqq p_r$; (b) $a_1\leqq4$, $a_i\leqq3$ for $1<i$, $r<p_1$. *R. Baer* (Urbana, Ill.).

Taketa, Kiyosi. Über die Existenz einer Untergruppe, deren Ordnung ein Produkt von zwei verschiedenen Primzahlpotenzen ist. Proc. Imp. Acad. Tokyo **19**, 609–610 (1943). [MF 14856]

If $\mathfrak{G}$ is any group whose order g is divisible by at least two primes p and q, then $\mathfrak{G}$ has a subgroup (possibly $\mathfrak{G}$ itself) whose order is the product of two prime powers. If $(g/p^a, p)=1$ then $\mathfrak{G}$ has either a normal subgroup of index p^a or a subgroup of order p^aq^b, $ab\neq0$, $(p, q)=1$.

J. S. Frame (East Lansing, Mich.).

Hadwiger, H. Bemerkung über eine spezielle Basis für die symmetrische und alternierende Gruppe. Tôhoku Math. J. **49**, 87–89 (1942). [MF 14698]

The author gives an explicit representation for each of the $n!$ elements X of the symmetric group $\mathfrak{S}_n$ in terms of the generators $A=(1, 2, \cdots, n)$ and $B=(2, 3, \cdots, n)$ by defining $T_k=A^{1-k}B^{k-1}=(1, 2, \cdots, k)$, $k=1, 2, \cdots, n$, and setting $X=T_1^{x_1}T_2^{x_2}\cdots T_n^{x_n}$, $0\leqq x_k<k$. It is easily proved by induction that each element is represented uniquely in this manner. The element X belongs to the alternating group if and only if the sum of the exponents with even subscripts is even. *J. S. Frame* (East Lansing, Mich.).

Baer, Reinhold. Representations of groups as quotient groups. I. Trans. Amer. Math. Soc. **58**, 295–347 (1945). [MF 14228]

If G and F are given groups, we have a representation of G as quotient group of F when we have a normal subgroup N of F such that G is isomorphic to F/N. Two representations F/N and V/M of the same group G are said to be related if there exists a homomorphism ϕ of F into V and a homomorphism ν of V into F such that ϕ and ν induce reciprocal isomorphisms between F/N and V/M. In this manner all representations of G as quotient groups are distributed into classes of related representations. It is one of the aims of the author to determine "invariants" of these classes. This is accomplished in the following way. Let F/N be a representation of G. Using the commutator calculus, the author constructs various families $F(i)$, $N(i)$ of subgroups of F such that (1) $F(i)$ is a fully invariant subgroup of F; (2) $N(i)$ is a normal subgroup of $F(i)$ which is fully invariant relative to N in the sense that every endomorphism of F which maps N into N maps $N(i)$ into $N(i)$; (3) the quotient group $F(i)/N(i)$ is an invariant of the class of representations to which F/N belongs. For this work, a detailed investigation of the derived series of a group relative to a normal subgroup is needed. One of the results obtained concerning related representations is a generalization of a theorem of H. Hopf [Comment. Math. Helv. **14**, 257–309 (1942); these Rev. **3**, 316].

A representation F/N of G is called a true representation when every automorphism of G is induced by an endomorphism of F; related true representations are said to be similar. Two groups F and V are termed similar if any two representations F/N and V/M of the same group G are

always similar representations. However, this relationship of similarity is not transitive and a group need not be similar to itself. A subgroup R of a group G is termed a retract of G if it is the image of G under an idempotent isomorphism of G. The retracts of similar self-similar groups are similar. Conversely, if Ξ is a set of similar and self-similar groups, then R is a retract of a group in Ξ if and only if R is a homomorphic image of a group of Ξ and is similar to every group in Ξ. Numerous further results are obtained concerning the concepts introduced.

It is shown, furthermore, that a group is a free group if and only if it is similar to every free group. If F is a free group and J is a fully invariant subgroup, then F/J is termed a reduced free group. Various properties of reduced free groups are established. If two reduced free groups R and S are written as quotients $F/N \cong R$ and $V/M \cong S$ of free groups F and V and if every homomorphism of F into V maps N into M and every homomorphism of V into F maps M into N, then F/N and V/M are similar groups, which are termed similarly reduced free groups. If Ξ is a system of similarly reduced free groups, all the representations of a given group G as quotient group of a group in Ξ belong to the same class of related true representations. This class may then be called an absolute class, since it depends only on G and the system Ξ. The invariants of such a class are, therefore, absolute invariants of G. Classes of similarly reduced free groups are characterized by the fact that the system of their homomorphic images is the class of all groups satisfying a suitable set of identical relations. A characterization of these systems of groups by properties of a lattice theoretical type is given.

R. Brauer (Toronto, Ont.).

Baer, Reinhold. Representations of groups as quotient groups. II. Minimal central chains of a group. Trans. Amer. Math. Soc. 58, 348–389 (1945). [MF 14229]

[Cf. the preceding review.] A normal subgroup N of a group G determines a minimal central chain ${}_iN$ which is defined inductively by the equations ${}_0N=N$, ${}_iN=(G, {}_{i-1}N)$, that is, ${}_iN$ is the group generated by the commutators of elements of G and ${}_{i-1}N$. In particular, if N is taken as the group G the lower central chain of G is obtained; for this, the notation ${}^iG$ will be used. Many results concerning these groups ${}_iN$ and ${}^iG$ are given. We quote here the following examples. If G/N is a finite group of order n, then ${}^iG/{}_iN$ is finite for every i and every prime number dividing the order of ${}^iG/{}_iN$ is a divisor of n. If N is a normal subgroup of the free group G, then ${}_iN/{}_{i+1}N$ is the direct product of $({}_iN \cap {}^{i+1}G)/{}_{i+1}N$ and of a free Abelian group. If the subgroup W of G contains ${}_{i+1}N$ and if ${}_iN/{}_{i+1}N$ is the direct product of $({}_iN \cap {}^{i+1}G)/{}_{i+1}N$ and of $W/{}_{i+1}N$, then the author calls ${}^iG/W$ an ith representation group of G/N. These groups are generalizations of the representation group introduced by I. Schur in connection with the investigation of the representation of groups of finite order by means of collineations; in Schur's case we have $i=0$. Every ith representation group of G/N is a central extension of $({}_iN \cap {}^{i+1}G)/{}_{i+1}N$ by ${}^iG/{}_iN$ and these two groups are invariants of G/N. As his second main result, the author establishes the fact that the manifold of all ith representation groups of G/N is in a certain sense a homomorphic image of the manifold of all the Abelian extensions of $({}_iN \cap {}^{i+1}G)/{}_{i+1}N$ by ${}^iG/({}_iN{}^{i+1}G)$. There are further results concerning group extensions which there is not space to quote here.

R. Brauer (Toronto, Ont.).

Baer, Reinhold. Representations of groups as quotient groups. III. Invariants of classes of related representations. Trans. Amer. Math. Soc. 58, 390–419 (1945). [MF 14230]

[Cf. the two preceding reviews.] The author derives criteria for two representations of a group to be related or to be equivalent, two representations F/N and V/M being equivalent if an isomorphic mapping of F on V exists which maps N on M. A fully invariant subgroup of a group G, the potence $P(G)$ of G, is introduced which is needed for some of the work. The group $P(G)$ is the join group of all the subgroups S of G which satisfy the relation $S=(S, G)$. Relations between the different invariants of classes of related representations studied in the first part are derived. Homomorphisms which preserve the multiplicator are investigated. The author is interested, in particular, in obtaining criteria for a homomorphism to be an isomorphism.

R. Brauer (Toronto, Ont.).

Shoda, Kenjiro. Bemerkungen über die induzierten Charaktere endlicher Gruppen. Proc. Imp. Acad. Tokyo 18, 336–338 (1942). [MF 14764]

Let G be a group of finite order and let H and H' be invariant subgroups of G. If χ is a character of H and χ' a character of H', we can form the two characters χ^* and χ'^* of G induced by the characters χ and χ'. The author gives necessary and sufficient conditions for the equality of χ^* and χ'^*.

R. Brauer (Toronto, Ont.).

Kondô, Kôiti. Decomposition of the characters of some groups. II. Proc. Phys.-Math. Soc. Japan (3) 23, 783–787 (1941). [MF 15008]

Following the procedure of part I [same Proc. (3) 23, 265–271 (1941); these Rev. 3, 196] and limiting attention to the subgroup $O(n-1)$ of the full real orthogonal group $O(n)$, the author obtains the decomposition of an irreducible representation of $O(n)$ into irreducible representations of $O(n-1)$. A similar reduction is given for the proper orthogonal group $O^+(n)$ and for the symplectic group $Sp(n)$.

G. de B. Robinson (Toronto, Ont.).

Suetuna, Z. Zur theorie der Gruppencharaktere. Jap. J. Math. 18, 729–744 (1943). [MF 14981]

Let $\mathfrak{G}$ be a group of finite order and $\mathfrak{H}$ a self-conjugate subgroup of index k such that $\mathfrak{G}/\mathfrak{H}$ is Abelian. A simple character χ of G will be compound for H; if ψ is a simple character of H contained in χ, we may write

$$\mathfrak{G}=\mathfrak{G}'+\mathfrak{G}'\tau_1+\cdots+\mathfrak{G}'\tau_{n-1},$$

where $\mathfrak{G}'$ is the totality of elements τ in $\mathfrak{G}$ such that $\psi(\sigma)=\psi(\tau\sigma\tau^{-1})$. Then $\chi'(\sigma)=m\psi(\sigma)$ in $\mathfrak{H}$, where m divides k. It follows from a paper by Clifford [Ann. of Math. (2) 38, 533–550 (1937)] that if $\mathfrak{G}/\mathfrak{H}$ is cyclical then $m=1$; the author gives a group-theoretic proof of this result. This theorem enables him to generalise previous conclusions [same J. 16, 63–69, 79–91 (1939); these Rev. 1, 258, 161] to the case where the factor group $\mathfrak{G}/\mathfrak{H}$ is linear mod p^n.

G. de B. Robinson (Toronto, Ont.).

Osima, Masaru. Note on the Kronecker product of representations of a group. Proc. Imp. Acad. Tokyo 17, 411–413 (1941). [MF 14721]

Let $\mathfrak{G}$ be a group of finite order and let R be its regular representation. If V is a representation of $\mathfrak{G}$ of degree m and if the underlying field has characteristic 0, then it is easily seen that the product representation $V \times R$ is similar

to the representation mR which is the direct sum of m constituents R. The author shows that the same result holds for modular fields. If $U_1, \cdots, U_k$ are the indecomposable parts of R, then the representations $V \times U_i$ split completely into representations U_j. This had been mentioned as a conjecture by the reviewer and C. Nesbitt [Ann. of Math. (2) **42**, 556–590 (1941); these Rev. **2**, 309]. The author shows further that, if R is the representation of $\mathfrak{G}$ induced by the 1-representation of a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, then $V \times R$ is similar to the representation of $\mathfrak{G}$ induced by the representation $V(\mathfrak{H})$ of $\mathfrak{H}$. *R. Brauer* (Toronto, Ont.).

Osima, Masaru. On primary decomposable group rings. Proc. Phys.-Math. Soc. Japan (3) **24**, 1–9 (1942). [MF 15017]

Let $\mathfrak{G}$ be a group of finite order $g=p^a g'$, $(p, g')=1$, and let Γ be the group ring of $\mathfrak{G}$ over any field K of characteristic p. The main result of this paper, an extension of a theorem of Brauer and Nesbitt [Ann. of Math. (2) **42**, 556–590 (1941); these Rev. **2**, 309] is that Γ is primary decomposable (that is, the direct sum of primary algebras) if and only if $\mathfrak{G}$ contains a normal subgroup $\mathfrak{G}'$ of order g'. Taking K so that the absolutely irreducible representations of $\mathfrak{G}$ may be written with coefficients in K, and using the methods of Brauer and Nesbitt and Nakayama [Ann. of Math. (2) **39**, 361–369 (1938)] the author goes on to study the modular representations of a group $\mathfrak{G}$ having such a subgroup $\mathfrak{G}'$ and the structure of its group-ring Γ. In particular, the representations of $\mathfrak{G}$ induced by irreducible representations of $\mathfrak{G}'$ are shown to be equivalent to the indecomposable constituents of the regular representation of $\mathfrak{G}$; an expression is obtained for the number of irreducible representations of $\mathfrak{G}$ whose degrees are exactly divisible by p^α, $0 \leqq \alpha \leqq a$; it is shown that the irreducible representations of $\mathfrak{G}$ may be induced by those of certain subgroups lying between $\mathfrak{G}'$ and $\mathfrak{G}$. The radical of Γ is characterized in terms of the radicals of the group rings of subgroups of a p-Sylow subgroup of G and Γ itself is shown to be uni-serial if and only if $\mathfrak{G}$ contains a cyclic p-Sylow subgroup. Finally, those groups whose ordinary irreducible representations remain irreducible as modular representations are determined.

S. A. Jennings (Vancouver, B. C.).

Golberg, P. Infinite semi-simple groups. Rec. Math. [Mat. Sbornik] N.S. **17**(59), 131–142 (1945). (Russian. English summary) [MF 14597]

Le produit direct d'un nombre arbitraire de groupes simples sera nommé un groupe complètement réducible. Envisageons tous les sous-groupes d'un groupe donné G, qui sont complètement réducibles et ne possèdent pas de centre. Parmi eux il y a un seul qui soit maximal. Si le centralisateur de ce sous-groupe se réduit à l'unité, G sera appelé semi-simple. L'absence de sous-groupes normaux abéliens de G est une condition nécessaire pour que G soit semi-simple. Cette condition est de même suffisante au cas des groupes finis, de sorte que la définition de l'auteur s'accorde avec la définition originale de Fitting [Jber. Deutsch. Math. Verein. **48**, 77–141 (1938)], si l'on se borne aux groupes finis. En général la condition ne suffit pas (voir par exemple les groupes libres d'un rang plus grand que l'unité), mais on peut admettre au lieu des groupes finis les groupes qui remplissent la condition suivante: chaque suite de sous-groupes de G, qui commence avec G et dont chacun est un sous-groupe normal du précédent, doit être fini. En se servant de sa définition de groupe semi-simple, l'auteur réussit à prononcer les résultats de Fitting regardant la structure des groupes semi-simples pour les groupes infinis.

H. Freudenthal (Amsterdam).

Grayev, M. On the theory of complete direct products of groups. Rec. Math. [Mat. Sbornik] N.S. **17**(59), 85–104 (1945). (Russian. English summary) [MF 14595]

Un ensemble infini de groupes étant donné, il est plus naturel de définir leur produit direct non de la manière ordinaire, mais en admettant les produits infinis des éléments respectifs: c'est le produit direct complet. L'auteur réunit ces deux définitions en une seule, plus générale. Soient donnés deux systèmes (infinis) de groupes, $A_n[A_n']$, $A_n' \subset A_n$. On forme le groupe A dont les éléments sont les produits infinis $\prod a_n$ ($a_n \epsilon A_n$ pour chaque n et $a_n \epsilon A_n'$ pour chaque n à un nombre fini près), et le groupe A' dont les éléments sont les produits infinis $\prod a_n'$ ($a_n' \epsilon A_n'$). Le paire de groupes $A[A']$ sera nommé le produit direct complet.

L'auteur étudie la décomposition d'un paire donné $G[G']$ en produit direct complet et il démontre des théorèmes d'unicité et de raffinement qui généralisent des théorèmes bien connus au cas des produits directs ordinaires. Enfin il donne une condition nécessaire et suffisante pour que le produit direct ordinaire soit facteur direct du produit direct complet: tous les groupes A_n', à un nombre fini près, doivent être commutatifs et de la forme $A_n' = B_n' \times C_n'$, où les C_n sont des groupes complets (c'est-à-dire admettant la division par un nombre naturel quelconque), et où les B_n' sont des groupes périodiques, les ordres des éléments étant uniformément bornés. *H. Freudenthal* (Amsterdam).

★Birkhoff, Garrett. Lattice-ordered Lie groups. Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 209–217, Füssli, Zürich, 1945.

A Lie tl-group is a Lie group which is a lattice in such a way that lattice operations are continuous and group translations are lattice isomorphisms. A Lie l-algebra is a Lie algebra with a lattice ordering such that the set of positive elements is invariant under inner automorphisms and multiplication by positive scalars. Principal results: every finite dimensional Lie l-algebra is solvable; the only Lie tl-groups are the powers of the additive group of real numbers.

P. A. Smith (New York, N. Y.).

Iwasawa, Kenkichi. On the structure of conditionally complete lattice-groups. Jap. J. Math. **18**, 777–789 (1943). [MF 14984]

The author proves the reviewer's conjecture [Ann. of Math. (2) **43**, 298–331 (1942), in particular, p. 329; these Rev. **4**, 3] that a conditionally complete l-group G is commutative. First he shows that, for any $a>0$, G is the direct union of the subgroup of b such that $a \cap |b| = 0$ and the subgroup of c with $|c| = \lim_{n\to\infty} na \cap |c|$. He then shows that $a^2 = b^2$ implies $a=b$, and that $ab^2 = b^2a$ implies $ab = ba$. Next he shows that G is the direct union of the subgroup S generated by "singular" elements, or elements $a \geqq 0$ such that $0<e<a$ implies $e \cap (a-e) = 0$, and the subgroup V of "vector" elements, or elements b such that λb exists for all dyadic fractions $\lambda = 2^{-n}$. It was known previously that S was commutative, essentially because disjoint elements are permutable; the author observes that S is a restricted power of the additive group of the integers. Using the spectral resolution of H. Freudenthal [Nederl. Akad. Wetensch., Proc. **39**, 641–651 (1936)] and H. Nakano [Jap. J. Math. **17**, 425–511 (1941); these Rev. **3**, 210], it is shown that V is a vector lattice. The argument assumes familiarity with

the subject; the essential point is that one can write $a=\int\lambda de_{\lambda,a}$ and $b=\int\mu de_{\mu,b}$, where $de_{\lambda,a}$ and $de_{\mu,b}$ are permutable. *G. Birkhoff* (Cambridge, Mass.).

Iwasawa, Kenkiti. On the structure of infinite *M*-groups. Jap. J. Math. 18, 709–728 (1943). [MF 14980]

A group is an M-group if the lattice of its subgroups is modular. Finite M-groups have already been characterized by the author [J. Fac. Sci. Imp. Univ. Tokyo Sect. I. **4**, 171–199 (1941); these Rev. **3**, 193] and the present paper extends these results to a wide class of infinite groups. It is shown that in any infinite M-group $\mathfrak{G}$ the elements of finite order form a characteristic subgroup $\mathfrak{E}$. If $\mathfrak{G}\neq\mathfrak{E}$ then both $\mathfrak{G}$ and $\mathfrak{E}$ are Abelian and $\mathfrak{G}$ itself is Abelian if $\mathfrak{G}/\mathfrak{E}$ is of rank at least 2. The only non-Abelian M-groups, therefore, which contain elements of infinite order are extensions of Abelian groups all of whose elements are of finite order by an infinite Abelian group of rank one; extensions of this type which yield M-groups are completely determined. In order to characterize M-groups without elements of infinite order, the author assumes that any group of finite length is necessarily of finite order. Subject to this restriction it is shown that M-groups all of whose elements are of finite order are very much like finite M-groups, in that they are direct products of groups of one or more of the following types: (i) Hamiltonian groups; (ii) modular p-groups, which if not Abelian are of the type $\{\mathfrak{A}, T\}$, where $\mathfrak{A}$ is an Abelian normal subgroup with elements of orders at most p^n, and $TAT^{-1}=A^{1+p^s}$ for all $A\epsilon\mathfrak{A}$, $T^{p^m}=A_0$, $A_0^{p^s}=1$, where $A_0\epsilon\mathfrak{A}$ and s depends only on $\mathfrak{A}$ and T subject to the condition $s+m\geqq n$ (and $s\geqq 2$ if $p=2$); (iii) modular pq-groups of the form $\{\mathfrak{P}, Q\}$, where $\mathfrak{P}$ is Abelian and such that $P^p=1$ for all $P\epsilon\mathfrak{P}$, Q of order a power of q, p and q being distinct primes such that $QPQ^{-1}=P^r$, $r\not\equiv 1 \bmod p$, $r^q\equiv 1 \bmod p$, r depending only on $\mathfrak{P}$ and Q. *S. A. Jennings.*

Inaba, Eizi. Über modulare Verbände, welche die Untergruppen einer endlichen abelschen Gruppe bilden. I. Proc. Imp. Acad. Tokyo 19, 528–532 (1943). [MF 14851]

The modular lattice $L(G)$ of all subgroups of a finite Abelian group G is "half-primary," in the sense that it is a direct union of "primary" lattices in which every interval a/b is a chain or contains no neutral elements. The author shows that these lattice-theoretic properties imply among other things that (1) every element is a join of elements in chain intervals $c/0$, (2) if e_v is the join of elements c_i such that $c_i/0$ is a chain of length v, then e_v/e_{v-1} is a complemented modular lattice. [Cf. R. Baer, Trans. Amer. Math. Soc. **52**, 283–343 (1942); these Rev. **4**, 109.]
G. Birkhoff (Cambridge, Mass.).

Kawada, Yukiyosi. Bemerkungen über das Weilsche Mass auf einer abelschen Gruppe. Proc. Imp. Acad. Tokyo **19**, 348–355 (1943). [MF 14830]

This paper outlines without detailed proofs the development of the following idea: if G is an abstract Abelian group endowed with an invariant measure in the sense of Weil, then the theory of the measurable characters of G can be based on an application of the Gelfand-Raikov-Segal methods to the normed ring of absolutely integrable functions on G, convolution playing the rôle of ring-multiplication; and, since the group X of such characters is locally compact and possesses an invariant measure in the sense of Haar, the measurable (=continuous) characters of X constitute a locally compact group $\bar{G}$ in which G appears as a dense subgroup and in which the invariant measure of Haar is correlated with the originally given measure in G in a natural manner. The development involves a number of clarifications of the central idea, as, for example, the fact that if N, a certain subgroup of G defined in terms of the characters in X, contains more than one element then it is G/N rather than G which is dense in $\bar{G}$. The development involves also some discussion of positive definite functions on G, and of unitary representations of G in the space of functions of absolutely integrable square on G. This paper offers a discussion of the topologization of groups with measure alternative to the original direct treatment due to Weil [C. R. Acad. Sci. Paris **202**, 1147–1149 (1936)].
M. H. Stone (Cambridge, Mass.).

Anzai, Hirotada. On compact topological rings. Proc. Imp. Acad. Tokyo **19**, 613–615 (1943). [MF 14858]

A compact topological ring R may be considered as a topological Abelian group with respect to its operation of addition and is thus seen to determine the associated character group G^*. It is noted that R may be continuously represented in the ring of endomorphisms θ of G^* (when the latter is suitably topologized) by virtue of the mapping $a\rightarrow\theta_a$, where the characters g^* and $\theta_a g^*$ are connected by the relation $g^*(xa)=\theta_a g^*(x)$ for all x in R. The following results are proved with the help of this representation: if R is connected, then $xy=0$ for all x, y in R; if R has no left (right) total zero divisor (that is, an element $a\neq 0$ such that $xa=0$ for all x) then R is totally disconnected; under the same condition R is a limit of finite rings. Further remarks are made in case R is taken to be locally compact. In particular, it is shown that, if R is connected and has no compact nilpotent ideal, then R is a hypercomplex number system over the real field [see N. Jacobson and O. Taussky, Proc. Nat. Acad. Sci. U. S. A. **21**, 106–108 (1935)].
M. H. Stone (Cambridge, Mass.).

Anzai, Hirotada, and Kakutani, Shizuo. Bohr compactifications of a locally compact Abelian group. I. Proc. Imp. Acad. Tokyo **19**, 476–480 (1943). [MF 14846]

A Bohr compactification of a locally compact Abelian group H is any Abelian group G containing as an everywhere dense subgroup a continuous homomorphic image of H. It is shown that any Bohr compactification of H can be obtained by completing a group G/N with respect to a uniform structure induced in a natural manner by a uniform structure superimposed on H, the latter being defined with the aid of a given family of almost periodic functions on H; N consists of those elements which are not separated from the zero of H by the uniform structure. The discrete character group G^* of G is algebraically isomorphic to a certain subgroup $X^*(F^*)$ of the character group H^* and the given topology of H has a countable basis if and only if X^* is countable. The case $X^*=H^*$ yields a universal Bohr compactification for H, namely, one which embraces, in a certain sense, all Bohr compactifications of H.
P. A. Smith (New York, N. Y.).

Anzai, Hirotada, and Kakutani, Shizuo. Bohr compactifications of a locally compact Abelian group. II. Proc. Imp. Acad. Tokyo **19**, 533–539 (1943). [MF 14852]

The results in part I [see the preceding review] yield the following facts about monothetic (solenoidal) groups, that is, topological Abelian groups with everywhere dense cyclic (one-parameter) subgroups. A compact Abelian G is monothetic (solenoidal) if and only if its discrete character group

is isomorphic to a subgroup of the additive group of reals mod 1 (reals) with discrete topology. [Nearly the same characterization of compact monothetic groups was given by Halmos and Samelson, Proc. Nat. Acad. Sci. U. S. A. **28**, 254–258 (1942); these Rev. **4**, 2.] Any compact monothetic (solenoidal) G can be obtained by completing the additive group of integers (additive group of reals) with respect to a uniform structure defined in a certain manner with the aid of a suitable family of almost periodic functions. Every compact solenoidal group G is monothetic. However, if G does not satisfy the second countability axiom, not every one-parameter subgroup which is dense in G need contain a cyclic subgroup which is dense in G.

P. A. Smith (New York, N. Y.).

Kakutani, Shizuo. On cardinal numbers related with a compact Abelian group. Proc. Imp. Acad. Tokyo **19**, 366–372 (1943). [MF 14833]

Let G be a compact Abelian group containing infinitely many elements. Let $\mathfrak{p}$ be the cardinal number of G, $\mathfrak{m}$ the cardinal number of the discrete character group of G. Then $\mathfrak{p}=2^{\mathfrak{m}}$. Moreover, $\mathfrak{m}$ is the smallest cardinal number of open sets forming a basis for G, and also the smallest number forming a basis at the zero element. The smallest number of elements forming an everywhere dense set in G is the smallest cardinal $\mathfrak{b}$ such that $2^{\mathfrak{b}}\geqq\mathfrak{m}$. *P. A. Smith.*

Garrison, G. N. Note on invariant complexes of a quasigroup. Ann. of Math. (2) **47**, 50–55 (1946). [MF 15658]

An invariant complex is a subset H such that for all a, b there is a c such that $(Ha)(Hb)=Hc$. The existence of such a set is equivalent to the existence of a homomorph. This concept was introduced by the author [same Ann. (2) **41**, 474–487 (1940); these Rev. **2**, 7]. It can happen that, of these cosets H, Ha, Hb, none, one, or more than one is a subquasigroup. An example is given in which the last is true. For finite quasigroups the Jordan-Hölder theorem is proved for those composition series terminating in the same idempotent element. The example shows that the last condition is essential. For the special cases of loops these results were already known [A. A. Albert, Trans. Amer. Math. Soc. **55**, 401–419 (1944); M. F. Smiley, Bull. Amer. Math. Soc. **50**, 782–786 (1944); these Rev. **6**, 42, 147].

H. H. Campaigne (Arlington, Va.).

Prenowitz, Walter. Descriptive geometries as multigroups. Trans. Amer. Math. Soc. **59**, 333–380 (1946). [MF 15654]

Descriptive geometries are characterized as commutative multigroups satisfying six additional postulates. This is done by defining $a+b$ to be the open line segment a to b. Convex sets are seen to be those closed under addition. The least convex set containing a given finite set of n independent points is seen to be an $(n-1)$-simplex, with the interior as the sum of all n points, and the sum of any fewer as part of the frontier. Many formal laws of manipulation are developed, some of the more important based on the symbol $\approx$, where $A\approx B$ means that the intersection $A\cap B$ of the sets A and B is not void. It is surprising that it is possible to get many laws analogous to those for equalities. The closest analogy to cosets is found in sets of the type $N-(N-a)=N-a'$ (for some a') which are half-spaces. That is, if N is a point, $N-(N-a)$ is an open ray; if N is a line, then $N-(N-a)$ is a half-plane. These half-spaces form a multigroup G/N which turns out to be a spherical space.

H. H. Campaigne (Arlington, Va.).

Tchounikhine, S. A. On the theory of non-associative n-groups satisfying postulate K. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 7–10 (1945). [MF 15225]

Let an $n$ary operation be defined in a set G so that to every sequence of elements $a_{m_1}\cdots a_{m_n}$ of G a "product" element $a_m=a_{m_1}\cdots a_{m_n}$ exists. Such a set G is said to be a nonassociative n-group if any element in the equation $a_m=a_{m_1}\cdots a_{m_n}$ is determined uniquely by the remaining n elements. The n-groups considered by the author satisfy an additional generalized associative law, "postulate K." This postulate requires that, for every product b of the form $b=a_{m_1}\cdots a_{m_{i-1}}(a_{m_i}\cdots a_{m_{i+n-1}})a_{m_{i+n}}\cdots a_{m_{2n-1}}$, and for every integer $i\leqq n$, there exists an integer $j\leqq n$ so that $b=a_{p_1}\cdots a_{p_{j-1}}(a_{p_j}\cdots a_{p_{j+n-1}})a_{p_{j+n}}\cdots a_{p_{2n-1}}$, where $(p_1\cdots p_{2n-1})$ is a permutation of the subscripts $(m_1\cdots m_{2n-1})$ which depends only upon i and j and not upon the particular group elements which make up the product b. (The first formula for the product b is misprinted in the original.) A symbolism for handling subsets of elements of such an n-group is described; theorems implying the existence of analogues of right and left units, and coset decompositions modulo a sub-n-group are stated. *S. A. Jennings.*

San Juan, R. A generalization of the concept of a group. Revista Mat. Hisp.-Amer. (4) **3**, 354–356 (1943). (Spanish) [MF 12176]

A "system" is defined to be a set of elements in which there is a nonassociative multivalued product ab such that the equations $ax=b$ and $ya=b$ have solutions. A number of definitions are given, for example, subsystem, homomorphism, Abelian system; one theorem, concerning a system homomorphic to a group, is proved. *H. S. Wall.*

ANALYSIS

Calculus

Busk, Thøger. A proof of Steffensen's generalization of Leibniz's theorem. Mat. Tidsskr. B. **1946**, 61–62 (1946). (Danish) [MF 16307]

A proof by induction of the formula

$$\varphi(x_0, x_1, \cdots, x_n)=\sum_{r=0}^{n} f(x_0, \cdots, x_r)g(x_r, \cdots, x_n),$$

where $\varphi(x)=f(x)g(x)$ and $F(y_0, y_1, \cdots, y_n)$ is the nth divided difference of $F(x)$ based on the points indicated.

R. P. Boas, Jr. (Providence, R. I.).

Weber, C. Potenzreihenzerlegung von $\operatorname{tg} x$ und $\operatorname{ctg} x$. Z. Angew. Math. Mech. **21**, 252–253 (1941). [MF 15858]

The author derives recurrence relations for the coefficients in the series in question. He is evidently unfamiliar with their expressions in terms of Bernoulli numbers.

R. P. Boas, Jr. (Providence, R. I.).

Leemann, W. Verschiedenartige Auflösungen einer Minimumsaufgabe. Schweiz. Z. Vermessgswes. Kulturtech. **41**, 232–235 (1943). [MF 15575]

The problem is to determine the dimensions of a rectangular building of given area so as to minimize the total cost of sidewalks, of different costs per unit length, along two of its sides.

Stewart, B. M., and Herzog, F. Cylinders in a cone. Bull. Amer. Math. Soc. **52**, 94–100 (1946). [MF 15446]

Let a cone be cut by n parallel hyperplanes and let V be the sum of the volumes of the cylinders formed on successive cross-sections. The authors solve the problem of maximizing V both with and without the restriction that the cylinders have equal volumes. *I. Kaplansky.*

Katô, Heizaemon. Graphic explanations of Kwankai-no-ki. Tôhoku Math. J. **49**, 53–59 (1942). (Japanese) [MF 14695]

This paper gives the calculation for the center of gravity of a semi-ellipsoidal shell of uniform thickness, whose inner surface is an ellipsoid of revolution and whose outer surface is generated by the end point of a normal of constant length moving over the inner surface. The problem is of historical interest. *C. Kikuchi* (East Lansing, Mich.).

Nair, U. S., and Mahajani, G. S. Generalisation of a certain definite integral. Math. Student **13**, 55–56 (1945). [MF 15644]

The integral discussed is

$$\int_0^\infty \prod_{k=1}^n \frac{\sin a_k x}{x} \prod_{l=1}^m \cos b_l x\, dx$$

with no restrictions on the (real) parameters a_k, b_l.
R. P. Boas, Jr. (Providence, R. I.).

Roure, Henri. Réduction aux fonctions Eulériennes de quelques types d'intégrales définies. Ann. Fac. Sci. Univ. Toulouse (4) **6**, 107–110 (1943). [MF 15174]

Integrals of the forms

$$\int_0^\infty e^{-mx}x^q f(x)dx, \quad \int_0^1 x^{p-1}(1-x)^{q-1}f(x)dx$$

are evaluated by expanding $f(x)$ into a power series.
H. Pollard (New Haven, Conn.).

Doole, H. P. A contour integral and first order expansion problem. Nat. Math. Mag. **20**, 79–85 (1945). [MF 15412]

The expression

$$\lim_{m\to\infty} \int_{C_m} e^{-pcz}(1-e^{-cz})^{-1}z^{-k}dz,$$

where C_m is the circle $|z| = 2(m+\frac{1}{2})/c$, is known, when $k=1$, to have the value $\frac{1}{2}$ at $p=0$, 0 when $0<p<1$, $-\frac{1}{2}$ at $p=1$. In this note the result is extended to arbitrary real or complex numbers p and any positive integer k. An application is made to the differential equation $y'+\lambda x^h y=0$, $h>0$.
H. Pollard (New Haven, Conn.).

Bongiovanni, Emilia. Sopra alcuni integrali doppi relativi ad un'area ellittica. Boll. Un. Mat. Ital. (2) **5**, 102–106 (1943). [MF 16094]

The author gives two methods for the calculation of the integral of $dS/(Rr)$ extended over an area S of an ellipse, R and r being the distances of dS from the foci. These methods are simpler than those given by F. Tisserand [Recueil Complémentaire d'Exercices sur le Calcul Infinitésimal, Gauthiers-Villars, Paris, 1896, p. 154]. The method is generalized to the calculation of the integral of $P(x, y)dS/(Rr)$ over the same area, where x, y are Cartesian coordinates and $P(x, y)$ is a polynomial. *I. Opatowski.*

Ridder, J. Über den Greenschen Satz in der Ebene. Nieuw Arch. Wiskunde (2) **21**, 28–32 (1941). [MF 15710]

It follows from a theorem of Banach [Fund. Math. **7**, 225–236 (1925), in particular, p. 229] that, if J is a rectifiable Jordan curve and ϵ a positive number, then the network of lines $x=m\epsilon$, $y=n\epsilon$ ($m, n=0, \pm1, \pm2, \cdots$) can be translated so as to have only a finite number of points in common with J. The author uses this fact to obtain a very short direct proof of a strong form of Green's theorem in the plane. *L. H. Loomis* (Cambridge, Mass.).

Hofmann, O. Neuer Beweis des Entwicklungssatzes der Vektoralgebra. Z. Angew. Math. Mech. **21**, 311–312 (1941). [MF 15860]

Graffi, Dario. Sul teorema della divergenza superficiale e sul calcolo delle azioni capillari. Boll. Un. Mat. Ital. (2) **4**, 8–12 (1942). [MF 16049]

The author gives a new proof of a formula of P. Burgatti involving surface integrals of grad, rot and div. As an application he calculates the resultant force and moment due to the action of capillary forces on a surface. His formulae are substantially identical with those of R. Serini [same Boll. (2) **3**, 207–210 (1941); these Rev. **3**, 93]. *I. Opatowski.*

Theory of Sets, Theory of Functions of Real Variables

★Vitali, G., e Sansone, G. Moderna Teoria delle Funzioni di Variabile Reale. 2d ed., vol. I. Consiglio Nazionale delle Ricerche. Monografie di Matematica Applicata. Nicola Zanichelli, Bologna, 1943. 194 pp.

The first edition, edited by Sansone from a posthumous manuscript of Vitali, appeared in 1935. The second edition is substantially the same as the first, somewhat amplified in a few places. It gives a clear and concise account, along classical lines, of the one-dimensional Lebesgue integral and the necessary preparatory topics from the theory of functions of a real variable. The five chapters deal, respectively, with sets and transfinite numbers; measure of linear sets; measurable functions, functions of bounded variation; integration of measurable functions, integration of series; differentiation of integrals and of functions of bounded variation.
R. P. Boas, Jr. (Providence, R. I.).

Kreweras, Germain. Extension d'un théorème sur les répartitions en classes. C. R. Acad. Sci. Paris **222**, 431–432 (1946). [MF 16020]

Let A and B be two partitions of a set E, each into n parts and let h be the smallest number such that r parts of A contain no more than $r+h$ parts of B, where $1\leqq r\leqq n$. Let k be the smallest number such that $n+k$ elements serve to represent both partitions. Then $h=k$. The proof, based on Zassenhaus's [Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937] for the case $h=k=0$, is given in full.
J. W. Tukey (Princeton, N. J.).

van Aardenne-Ehrenfest, T. Proof of the impossibility of a just distribution of an infinite sequence of points over an interval. Nederl. Akad. Wetensch., Proc. **48**, 266–271 = Indagationes Math. **7**, 71–76 (1945). [MF 15798]

Let $\{a_n\}$ be an infinite sequence in an interval I; α, β are subintervals of I, $A_n(\alpha)$ is the number of points $a_m \subset \alpha$ with $m\leqq n$. The sequence $\{a_n\}$ is called "just" if, for some con-

stant C, $|A_n(\alpha)-A_n(\beta)|\leqq C$ whenever $|\alpha|=|\beta|$. The result is as indicated in the title, slightly extended.

H. D. Ursell (Leeds).

Erdös, Paul. On the Hausdorff dimension of some sets in Euclidean space. Bull. Amer. Math. Soc. **52**, 107–109 (1946). [MF 15448]

Soit E un ensemble fermé de l'espace euclidien E_n, x un point quelconque de E_n, $\Phi(x)$ l'ensemble des points de E dont la distance à x, soit $g(x)$, est minimale, $L(x)$ la plus petite variété linéaire (éventuellement un point) contenant $\Phi(x)$, $\lambda(x)$ la dimension de $L(x)$. S'appuyant sur un théorème de Roger-Saks, l'auteur démontre que (1) l'ensemble E_k des points x où $\lambda(x)>k-1$ $(k=1,\cdots,n)$ est inclus dans la réunion d'une infinité dénombrable de surfaces de mesure $(n-k+1)$-dimensionnelle nulle, et l'ensemble des points x où $\lambda(x)=n$ est dénombrable; (2) presque partout sur E, $g(x)$ a une dérivée nulle dans toute direction. [Remarques. Le raisonnement par lequel le référent obtient (1) pour $n=2$ [Rev. Sci. (Rev. Rose Illus.) **77**, 493–496 (1939); ces Rev. **1**, 109] est valable pour n quelconque car il s'appuye sur une proposition de M. Bouligand selon laquelle les propriétés du premier ordre de la fonction $g(x)$ au point x ne font intervenir que $\Phi(x)$. D'ailleurs (1) peut s'établir élémentairement en prouvant que le paratingent de l'ensemble $E[\lambda(x)=\lambda_0]$ laisse échapper les directions de $L(x)$.]

C. Pauc (Paris).

Beesley, E. M., and Morse, A. P. φ-Cantorian functions and their convex moduli. Duke Math. J. **12**, 585–619 (1945). [MF 15504]

Here $\varphi(0)=0$, $\varphi(x)$ is continuous and increasing in $x\geqq 0$; $\varphi^*(A)$ is the φ-measure of a linear set A [Hausdorff, Math. Ann. **79**, 157–179 (1918)]; I, J, K denote intervals, $|I|$ the length of I; H_n is a countable family of nonoverlapping subintervals of I, each $J\epsilon H_{n+1}$ a subinterval of some $K\epsilon H_n$; $\sigma(H_n)$ is the point-set sum of intervals of H_n and $A=\prod\sigma(H_n)$. If, for each n,

$$\sum_{J\epsilon H_n}\varphi(|J|)=\varphi(|I|),\qquad \sum_{J\epsilon H_n,\, J\subset K}\varphi(|J|)\leqq\varphi(|K|)$$

for every K, then it is proved that $\varphi^*(A)=\varphi(|I|)$ and $\varphi^*(KA)\leqq\varphi(|K|)$ for every K. Hereafter φ is required to be also convex upward; the above set A is then called φ-Cantorian and $f(x)$, the φ-measure of the part of A to the left of x, is called a φ-Cantorian function. With φ convex, a suitable H_{n+1} can always be constructed, given H_n. The least function $\Phi(t)$ convex upward in $t\geqq 0$ and such that $|f(x+t)-f(x)|\leqq\Phi(t)$ for all x, t is called the convex modulus of f. It is proved that $\Phi\leqq\varphi$ and a lower bound for Φ is also obtained. In many cases Φ is shown to coincide with that lower bound. Sets of points x in whose decimal representation the digits are restricted are discussed. If $L_n>0$, $\sum L_n<|I|$, a set $A\subset I$ is constructed of Lebesgue measure zero which cannot be covered by intervals J_n of lengths L_n.

Finally, a particular φ, a perfect set of dimension φ on the x-axis and its Lefschetz product with a congruent set on the y-axis are taken. A shear or rotation of this plane set gives a set which is not a countable sum of cross-products of sets of finite φ-measure, thus throwing light on the question of product measure.

H. D. Ursell (Leeds).

Kline, S. A. The representation of Baire's classes by transfinite sums of continuous functions. J. London Math. Soc. **20**, 4–7 (1945). [MF 15927]

According to the main result of this paper, a Baire function $f(x)$ of class α may be represented as the sum of a transfinitely convergent series of type ω^α of continuous functions. The author was not aware that this result had already been proved by M. Lavrentieff [Fund. Math. **5**, 123–129 (1924)].

A. Rosenthal (Albuquerque, N. M.).

Hadwiger, H. Bemerkung über additive Mengenfunktionale. Experientia **1**, 274–275 (1945). [MF 15390]

Let a family $\mathfrak{M}$ of closed sets in Euclidean n-space contain the unit cube and with A and B also $A+B$, $A\cdot B$, A^T and λA. Here T denotes any translation and λA is the set of all the points λx with $x\epsilon A$. In $\mathfrak{M}$ a real-valued functional $\varphi(A)$ is defined with $\varphi(0)=0$, $\varphi(A^T)=\varphi(A)$, $\varphi(A+B)+\varphi(A\cdot B)=\varphi(A)+\varphi(B)$; finally, $\varphi(A)$ is assumed to satisfy a certain continuity condition. Then for any given $A\epsilon\mathfrak{M}$, $\varphi(\lambda A)$ is a polynomial of degree at most n in λ.

P. Scherk (Saskatoon, Sask.).

Rosenthal, Arthur. On interval-functions and associated set-functions. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 153–156 (1945). [MF 13725]

This paper is a report of some results to appear in the forthcoming Hahn-Rosenthal book on set functions (written on the basis of a part of the manuscript left by Hahn for the second volume of his revised "Reelle Funktionen"). The results are concerned with a set function φ which in a certain sense is associated with a given interval function χ (with respect to a given totally additive set function ψ). Necessary and sufficient conditions for the existence of φ are formulated; in these conditions the derivative $D(\chi,\psi)$ of χ with respect to ψ plays an essential role. If the conditions are satisfied, φ can be represented by the Radon integral of that derivative,

$$\varphi(M)=\int_M D(\chi,\psi)d\psi.$$

J. F. Randolph (Oberlin, Ohio).

Ridder, J. Der Bairesche Satz bei Intervallfunktionen. Nieuw Arch. Wiskunde (2) **21**, 57–58 (1941). [MF 15712]

Closed intervals in R_n form a complete metric space, with the natural metric $\rho(I_1,I_2)=\max\{|a_i^1-a_i^2|,\ |b_i^1-b_i^2|\}$, where $I_j=\langle a_i^j, b_i^j\rangle$; hence, in particular, the Baire theorem on pointwise discontinuous functions may be applied to (real) functions of intervals.

A. J. Ward.

Ridder, J. Über k-fache approximative Differentiation von Reihen. Nieuw Arch. Wiskunde (2) **21**, 25–27 (1941). [MF 15709]

Let $F(x)=\sum_n f_n(x)$ be a convergent series of functions such that the series $\sum_n AD^{(k)}f_n(x)$ of kth approximate derivatives exists and converges uniformly for $a\leqq x\leqq b$. Then, for almost all n, $AD^{(k)}f_n(x)$ is in fact an ordinary kth derivative (for all x); it follows that $AD^{(k)}F(x)=\sum_n AD^{(k)}f_n(x)$.

A. J. Ward (Cambridge, England).

Ridder, J. Ueber Halbtangenten an Punktmengen. Nieuw Arch. Wiskunde (2) **21**, 168–193 (1943). [MF 15702]

The author gives new proofs and extensions of theorems (all of which are generalisations of the Denjoy theorems on derivatives of functions of one real variable) due to Verčenko and Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **2** (**1934 I**), 105–107; **5** (**1934 IV**), 361–364 (1934)], Roger [Acta Math. **69**, 99–133 (1938)], Haslam-Jones [Quart. J. Math., Oxford Ser. **3**, 120–132 (1932); **7**, 116–123, 184–190 (1936)] and Ward [Proc. London Math. Soc. (2) **39**, 339–362 (1935)]. He establishes first the Denjoy properties of

derivates for many-valued functions of n variables and applies these results to prove the theorems of Verčenko-Kolmogoroff and of Roger concerning the distribution of "half-tangents" in a general set in Euclidean space of n dimensions. He employs a covering theorem simpler than that of Vitali and also avoids the use of the differentiability almost everywhere of monotone functions of one variable.
U. S. Haslam-Jones (Oxford).

Myshkis, A. On the existence of the total differential on the boundary of a plane domain. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 82–85 (1945). [MF 15221]

Let G be an open plane set, A a point of $\bar{G}-G$, $f(B)$ a point-function defined on $G+A$, continuous at A, and such that its four x-derivates (y-derivates) have a common limit a (or b, respectively) as $B\to A$, B in G. It is shown that these conditions are sufficient for f to have the total differential $adx+bdy$ at A (over $G+A$) if and only if

$$\overline{\lim_{B\to A}}\ \underline{\lim_{C\to A}}\ \{l(\Gamma)/r(B,C)\}<\infty,$$

where $l(\Gamma)$ is the length of any rectifiable curve Γ joining B and C in G and $r(B,C)$ is the Euclidean distance. The author further considers the possible structure of $D=G\bar{G}_1\bar{G}_2$ when f is continuous on $G(\bar{G}_1+\bar{G}_2)$ and has, at each point of D, nonidentical total differentials over $\bar{G}_1$ and over $\bar{G}_2$, distinguishing certain cases, for example, according to the continuity and/or uniqueness of the differentials. [The problem is cited as Petrovsky's but no source is given.]
A. J. Ward (Cambridge, England).

Theory of Functions of Complex Variables

Shimizu, Tatsujiro. A condition for a function of a complex variable to be regular. Nippon Sugaku Buturigakkwai Zasshi 16, 139–140 = Coll. Papers Fac. Sci. Osaka Imp. Univ. Ser. A. 10, no. 13, 2 pp. (1942). (Japanese) [MF 15807]

The author defines a "directional derivative of order n at a point" by $d^nf/dz^n=(\bar{D}+De^{-2i\theta})^nf$, where $z=re^{i\theta}$, $D=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$. He gives the following necessary and sufficient condition for $f=u+iv$ to be regular in a domain: all directional derivatives of all orders exist at a point of the domain and

$$\left|\frac{\partial^n u}{\partial x^{n-j}\partial y^j}\right|,\quad \left|\frac{\partial^n v}{\partial x^{n-j}\partial y^j}\right|<\eta n!K^n,\quad \begin{matrix} j=0,1,\cdots,n;\\ n=0,1,2,\cdots,\end{matrix}$$

throughout the domain. *R. P. Boas, Jr.*

Salem, R., and Zygmund, A. Lacunary power series and Peano curves. Duke Math. J. 12, 569–578 (1945). [MF 15502]

The image of $|z|=1$ by the transform $w=f(z)$ is discussed when $f(z)=\sum a_kz^{n_k}$ $(n_{k+1}/n_k\geqq\lambda>1)$ is a lacunary power series with convergent $\sum|a_k|$. It is proved that this image is a Peano curve (fills a square) whenever $\lambda>\lambda_0$, where λ_0 is an absolute constant, provided that $\sum|a_k|$ converges slowly enough. The condition

$$(\lambda|a_1|+\lambda^2|a_2|+\cdots+\lambda^p|a_p|)/\lambda^p<c(\lambda)\sum_{p+1}^{\infty}|a_k|,$$

$$0<c(\lambda)<\frac{\lambda(\lambda-1)-2^{3/2}\pi(5\lambda-1)}{\lambda(\lambda-1)+2^{3/2}\pi(5\lambda-1)}$$

is sufficient. The question whether $\lambda_0=1$ or $\lambda_0>1$ remains open. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Okada, Yoshitomo. On interpolation by polynomials. Tôhoku Math. J. 48, 68–70 (1941). [MF 16349]

Walsh has shown that, if $f(z)$ is analytic in $|z|<\rho$, $\rho>1$, but not in any larger circle, and if $p_n(z)$ is the unique polynomial of degree n interpolating to $f(z)$ in the $(n+1)$th roots of unity, then $p_n(z)\to f(z)$ uniformly in any circle $|z|\leqq\rho_1<\rho$, while $\{p_n(z)\}$ diverges outside $|z|=\rho$ [cf. Bull. Amer. Math. Soc. 42, 715–719 (1936)]. The author replaces the roots of $z^{n+1}=1$ by the roots of $z^{n+1}+a_1^{(n)}z^n+\cdots+a_{n+1}^{n}=0$, assumed distinct and interior to $|z|\leqq1$, under the condition that, for some a in $0<a<\rho$ and some $A<1$, $\sum_{k=1}^{n+1}a^{-k}|a_k^{(n)}|\leqq A$.
R. P. Boas, Jr. (Providence, R. I.).

Whittaker, J. M. Representation of functions by series of polynomials. Proc. Math. Phys. Soc. Egypt 2, no. 3, 13–22 (1944). [MF 16336]

Let $p_n(z)$ be a basic set of polynomials [see the author's Interpolatory Function Theory, Cambridge University Press, 1935], that is, a set such that every polynomial can be expressed in one and only one way as a finite linear combination of them. Every function analytic at $z=0$ has a formal expansion ("basic series") in terms of the $p_n(z)$. Let $M_k(R)=\max|p_k(z)|$ for $|z|=R$, let $\{\pi_{nk}\}$ be the coefficients of the expansion of z^n in terms of $\{p_k\}$, and define $\omega_n(R)=\sum_k|\pi_{nk}|M_k(R)$, $\lambda(R)=\lim\sup_{n\to\infty}\{\omega_n(R)\}^{1/n}$. Let N_n be the number of nonzero π_{ni}. The author considers sets ("Cannon sets") for which $N_n^{1/n}\to1$ as $n\to\infty$. For such sets, Cannon [Proc. London Math. Soc. (2) 43, 348–365 (1937)] showed that the basic series is effective on $|z|=R$, that is, represents in $|z|\leqq R$ every function analytic in $|z|\leqq R$, if and only if $\lambda(R)=R$. By showing that $\lambda(R)$ satisfies an inequality of the form of Hadamard's three-circles theorem, the author shows that the circles on which a Cannon set is effective form an annulus. Furthermore, a Cannon series represents in $|z|<R$ every function analytic in $|z|<R$ if and only if $\lambda(r)<R$ for $r<R$, and represents near $|z|=0$ every function analytic at 0 if and only if $\lambda(0+)=0$.
R. P. Boas, Jr. (Providence, R. I.).

Denjoy, Arnaud. Sur les séries de fractions rationnelles. C. R. Acad. Sci. Paris 222, 709–712 (1946). [MF 16171]

Let Γ be a simple closed curve, formed by a finite number of arcs possessing a tangent varying continuously, and drawn in the positive sense relative to its finite region denoted by Γ^+. Let $F(x)$ be a function holomorphic in Γ^+ and on Γ, $\Re$ a region containing Γ^+ and Γ and in which $F(x)$ is analytic and uniform. Let $S(\Gamma^+)$ designate a series of rational fractions $\sum A_n/(x-a_n)$ for which $\sum|A_n|<\infty$ with poles a_n in Γ^+ and points of accumulation on Γ. Using the method of J. Wolff [same C. R. 173, 1327–1328 (1921)] the author gives a proof of the following theorem. Let σ and σ' be any two arc segments of Γ having no points in common. Then it is possible to construct a series $S(\Gamma^+)$ in such a way that its sum $G(x)$ may be continued into the region $\Re$ diminished by the cuts σ and σ' and such that every direct rotation by x around σ or σ' increases $G(x)$ by the period $F(x)$ or $-F(x)$, respectively. *M. S. Robertson.*

Mandelbrojt, Szolem. Sur les fonctions holomorphes et bornées dans une partie infinie d'un demi-plan. C. R. Acad. Sci. Paris 222, 361–363 (1946). [MF 16012]

The author indicates a proof of the following result. Let D be a closed domain in the $s=\sigma+it$ plane defined by

$\sigma \geqq d$, $|t| \leqq g(\sigma)$, where $g(\sigma)$ is positive and increasing for $\sigma \geqq d$, $g(\infty) = \pi/2$ and the area of the part D' of $|t| \leqq \pi/2$, $\sigma \geqq d$, not in D, is finite. Let $F(s)$ be not identically zero, analytic and bounded in D. If $M(\sigma) = \max |F(\sigma+it)|$ for $\sigma \geqq d$, $|t| \leqq g(\sigma)$, then $\int_d^\infty \log M(\sigma) e^{-\sigma} d\sigma > -\infty$. It is shown that for a special $g(\sigma)$ the condition that D' has finite area is essential. [The author wishes to point out that the example $\exp(-e^s)$ shows this for any $g(\sigma)$.] For an application, see the following review. *R. P. Boas, Jr.*

Mandelbrojt, Szolem. L'évaluation des coefficients d'une représentation asymptotique générale. C. R. Acad. Sci. Paris **222**, 471–473 (1946). [MF 16025]

The author states a theorem, more general than similar results which he gave earlier [Trans. Amer. Math. Soc. **55**, 96–131 (1944); these Rev. **5**, 176], giving an estimate for the coefficients in exponential sums which represent a function asymptotically in certain domains. *R. P. Boas, Jr.*

Wolff, J. Inégalités remplies par les fonctions univalentes. Nederl. Akad. Wetensch., Proc. **44**, 956–963 (1941). [MF 15763]

Let $f(z) = f(x+iy)$ be holomorphic, bounded and univalent in the half-plane $D(x>0)$ and let $\iint |f'|^2 dxdy < \infty$. The author gives a simple proof that $xf'(z) \to 0$ as $x \to 0$. On the line $x = c > 0$ the length $L(y)$ of the image of the segment $[c, c+yi]$ is $o(|y|^{\frac{1}{2}})$ as $|y| \to \infty$. For $p>0$, $\alpha>0$ consider the curves Γ_t in D whose equations are $y = \alpha x^{-p} + t$, $-\infty < t < \infty$. Then the length $L(z)$ of the image of the arc of Γ_t between an initial point z_0 and a point z whose abscissa $x \to 0$ satisfies $L(z) = o(|z|^{1+1/(2p)})$. For almost all points it on the imaginary axis, $f'(z) = o(|y-t|^{\frac{1}{2}}/x)$ when z tends towards it on a curve convex towards the left [see also Wolff, same Proc. **45**, 574–577 (1942); Comment. Math. Helv. **15**, 296–298 (1943); these Rev. **5**, 259, 234]. Analogous results are also obtained for functions $\phi(z)$ holomorphic, bounded and univalent in the unit circle $|z| < 1$.

M. S. Robertson (New Brunswick, N. J.).

Robertson, M. S. Star center points of multivalent functions. Duke Math. J. **12**, 669–684 (1945). [MF 15510]

Inequalities involving the coefficients a_n of a power series $f(z) = \sum_1^\infty a_k z^k$, p-valent in $|z| < 1$, are given under the assumption that $f(z)$ possesses a star centre. The point ζ, $|\zeta| < 1$, is said to be a star centre of $f(z)$ with respect to the circle $|z| = r < 1$ if $\arg(f(z) - f(\zeta))$ increases when $\arg z$ increases, where $|z| = r$. This is equivalent to

$$\Re z f'(z)/\{f(z) - f(\zeta)\} \geqq 0, \qquad |z| = r.$$

The point ζ is said to be a star centre of $f(z)$ (with respect to $|z| = 1$) if it is a star centre with respect to $|z| = r$ for all sufficiently large $r < 1$. It is clear that, if f is p-valent and a star centre ζ_1 exists, then there are p such centres ζ_s, $1 \leqq s \leqq p$, all of which give the same value $w_1 = f(\zeta_1)$. Some or all of the ζ_s may coincide (w_1 a multiple value). Of the various inequalities obtained we mention the following (it is assumed that $\zeta_s \neq 0$):

$$\left| \sum_1^p ((1+|\zeta_s|^2)/\zeta_s) - a_1/w_1 \right| \leqq 2p,$$

$$\left| \sum_1^p ((1+|\zeta_s|^4)/\zeta_s^2) - 2a_2/w_1 - a_1^2/w_1^2 \right| \leqq 2p.$$

There are similar inequalities involving general a_n; all of them are best possible. If $\zeta_1 \neq 0$, $a_1 = 1$ and $p = 1$ ($f(z)$ univalent), then

$$|a_2| \leqq \cos\{\arg(w_1/\zeta_1)\} + \tfrac{1}{2}|w_1^{-1} - (1+|\zeta_1|^2)/\zeta_1| \leqq 2,$$

$$|a_3| \leqq \tfrac{1}{3} + \tfrac{2}{3}\cos\{\arg(w_1/\zeta_1)\}\{1 + |w_1^{-1} - (1+|\zeta_1|^2)/\zeta_1|\} + \tfrac{1}{3}|w_1^{-1} - (1+|\zeta_1|^2)/\zeta_1|^2 \leqq 3.$$

Equality holds for $f(z) = z(1-z)^{-2}$, if $0 < \zeta_1 < 1$. There are also inequalities covering the case where some of the ζ_s vanish. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Bloch, André. Théorèmes d'algèbre et de géométrie. C. R. Acad. Sci. Paris **219**, 301–303 (1944). [MF 15259]

This note states a number of miscellaneous theorems dealing with algebraic curves, quadratic binary forms, the periods of Abelian integrals, and bounded analytic functions. *M. H. Heins* (Cambridge, Mass.).

Ferrand, Jacqueline. Sur les conditions d'existence d'une dérivée angulaire dans la représentation conforme. C. R. Acad. Sci. Paris **219**, 507–508 (1944). [MF 15283]

A necessary condition and two sufficient conditions are given for the existence of an angular derivative for a one-to-one directly conformal mapping of a simply-connected region whose boundary does not reduce to a single point. These results are related to the author's thesis [Ann. Sci. École Norm. Sup. (3) **59**, 43–106 (1942); these Rev. **6**, 207].

M. H. Heins (Cambridge, Mass.).

Tsuji, Masatsugu. On an extension of Bloch's theorem. Proc. Imp. Acad. Tokyo **18**, 170–171 (1942). [MF 14749]

The extension of Bloch's theorem to meromorphic functions is established by a proof which is a modification of the one due to Landau [Math. Z. **30**, 608–634 (1929)] for the original Bloch theorem. *M. H. Heins.*

Kunugui, Kinjiro. Sur une constante de la transformation conforme. Proc. Imp. Acad. Tokyo **19**, 278–281 (1943). [MF 14825]

The following theorem is due to Teichmüller [Deutsche Math. **2**, 96–107 (1937)]. Let $\zeta(w)$ be analytic and univalent for $|w| < 1$. Furthermore, let γ denote a rectilinear transversal of the image region. Then

$$(1) \qquad \int_\gamma \log |1/w(\zeta)|\,|d\zeta| \leqq cl,$$

where $w(\zeta)$ is the inverse of $\zeta(w)$ and c is a universal constant. Teichmüller conjectured that the best value of c is $\pi/2$ (attained by $\zeta(w) \equiv w/(1+w^2)$). The truth of this conjecture is established in the present note. The proof is based upon Koebe's distortion theorem, which implies the differential inequality

$$(2) \qquad |dw|/(1-|w|^2) \geqq |d\zeta|/|1-4\zeta^2|$$

if the endpoints of γ are normalized to be $-\frac{1}{2}$ and $+\frac{1}{2}$. Upon integration (2) yields a lower bound for $|w(\zeta)|$ on γ, which in turn gives the desired appraisal for the left hand side of (1). *M. H. Heins* (Cambridge, Mass.).

Dufresnoy, Jacques. Remarques complémentaires sur deux propriétés de la représentation conforme. Bull. Sci. Math. (2) **69**, 117–121 (1945). [MF 15887]

Some remarks on a paper of the author [same Bull. (2) **69**, 21–36 (1945); these Rev. **7**, 56] concerning the conformal mapping of simply-connected regions.

M. H. Heins (Cambridge, Mass.).

Dufresnoy, Jacques. Sur les fonctions méromorphes dans le cercle unité et couvrant une aire bornée. C. R. Acad. Sci. Paris **219**, 274–276 (1944). [MF 15257]

The following theorem is proved. Let $h(t)$ denote a function which is defined, monotone increasing and positive for $t>0$, which tends to zero with t and satisfies

$$\int_0^a t^{-1}\{h(t)/\log(1/t)\}^{\frac{1}{2}}dt<+\infty, \qquad 0<a<1.$$

If $f(z)$ is analytic for $|z|<1$ and is such that the Riemannian image of $(|z-1|<t)\cdot(|z|<1)$ with respect to $f(z)$ has an area less than $h(t)$, then the image of $0\leqq x<1$ (x real) with respect to $f(z)$ has finite length. The proof is based upon the following lemma. If $w=f(z)$ is analytic for $|z|<1$ and is such that the area of the Riemannian image of $|z|<1$ with respect to $f(z)$ is equal to s (finite), then the length l of the image of the segment $(-r, +r)$ (r positive, $r<1$) with respect to $f(z)$ satisfies $l<k\{s\log(1+r)/(1-r)\}^{\frac{1}{2}}$, where k is a universal constant. Extensions to meromorphic functions are indicated. *M. H. Heins* (Cambridge, Mass.).

Kametani, Shunji. The exceptional values of functions with the set of capacity zero of essential singularities. Proc. Imp. Acad. Tokyo **17**, 429–433 (1941). [MF 14723]

It is shown that, if $w=f(z)$ is a single-valued nonconstant analytic function defined in an open region D except for a bounded set $E\subset D$ of essential singularities of capacity zero, then $f(z)$ attains all finite values, except for those belonging possibly to a set of capacity zero, infinitely often in an arbitrary neighborhood of any point of E. This generalizes a result due to R. Nevanlinna [Eindeutige Analytische Funktionen, Springer, Berlin, 1936, p. 135]. The proof is based principally on Myrberg's work [Acta Math. **61**, 39–79 (1933)] concerning the existence of a Green's function on a given Riemann surface. *M. H. Heins.*

Shah, S. M. On proximate orders of integral functions. Bull. Amer. Math. Soc. **52**, 326–328 (1946). [MF 16199]

The author gives an elementary construction for Lindelöf proximate orders for entire functions of finite order. *R. P. Boas, Jr.* (Providence, R. I.).

Myrberg, P. J. Über den Fundamentalbereich der automorphen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 2, 25 pp. (1941). [MF 15209]

Let (a) be a denumerably infinite set of points lying in $|x|<R(\leqq+\infty)$ and having the properties that no limit point of (a) lies in $|x|<R$ and that every point of $|x|=R$ (if R is finite) is a limit point of (a). Furthermore, let $x(z)$ denote a uniformization mapping of $|z|<1$ onto $(|x|<R)-(a)$ (the ramification indices of the points of (a) being infinite). The function $x(z)$ is automorphic with respect to a Fuchsoid group. The properties of fundamental domains of this group and of their images with respect to $x(z)$ are investigated. The harmonic measure of the set of vertices of a fundamental domain defined with reference to the given fundamental domain and the corresponding harmonic function defined in the image of the fundamental domain with respect to $x(z)$ are treated. In the case where $R=+\infty$, this harmonic measure is always zero. In the case where $R<+\infty$, a sufficient condition is given which guarantees the harmonic measure to be nonnull for certain classes of fundamental domains. Examples are constructed for sets (a) in the case $R<+\infty$ for which the harmonic measure of the sets of vertices of any fundamental domain with respect to that domain is zero. With the aid of these examples it is concluded that there exist bounded analytic functions which attain their Fatou boundary values on a set of measure zero. The proofs are based largely upon the use of elementary harmonic majorants. *M. H. Heins.*

Roure, Henri. Sur une nouvelle classe de fonctions. Ann. Fac. Sci. Univ. Toulouse (4) **6**, 15–31 (1943). [MF 15172]

Simple generalization of Picard's results on automorphic functions of two variables [Acta Math. **1**, 297–320 (1882)]. *C. L. Siegel* (Princeton, N. J.).

Roure, Henri. Sur une nouvelle classe de fonctions. II. Ann. Fac. Sci. Univ. Toulouse (4) **7**, 99–122 (1945). [MF 16178]

In the paper reviewed above the author introduced a generalization of Picard's automorphic functions of several variables. His functions are defined by Poincaré series of 4 variables and he now attempts to prove that 5 such functions are always connected by an algebraic equation with constant coefficients. It is well known that the corresponding part of Picard's work presents serious gaps, but the author uses the same incomplete method without mentioning the researches of O. Blumenthal [Math. Ann. **56**, 509–548 (1903); **58**, 497–527 (1904)] which provide a safe foundation for theorems of this type. *C. L. Siegel* (Princeton, N. J.).

Fueter, Rud. Über die Quaternionenmultiplikation regulärer vierfachperiodischer Funktionen. Experientia **1**, 57 (1945). [MF 15386]

In dieser vorläufigen Mitteilung gibt der Verfasser bekannt, dass die "Theorie der komplexen Multiplikation der elliptischen Funktionen weitgehend auf den Bereich der vierfachperiodischen Funktionen übertragen" werden kann. Er geht dabei von Ergebnissen einer früheren Arbeit aus [Monatsh. Math. Phys. **48**, 161–169 (1939); diese Rev. **1**, 115], worin er die Darstellung jeder beliebigen vierfachperiodischen rechtsregulären Funktion angibt, zunächst unter einer gewissen Voraussetzung, die dann später von W. Nef bewiesen wurde [Comment. Math. Helv. **16**, 215–241 (1944); diese Rev. **5**, 241]. *P. Thullen* (Quito).

Sugawara, Masao. On the general Schwarzian lemma. Proc. Imp. Acad. Tokyo **17**, 483–488 (1941). [MF 14732]

Morita, Kiiti. Analytical characterization of displacements in general Poincaré space. Proc. Imp. Acad. Tokyo **17**, 489–494 (1941). [MF 14733]

Let R be the space of all $m\times n$ complex matrices Z with $|Z|\leqq 1$. Suppose that $f(Z)$ is a regular analytic function on R. The first author proves the following generalization of Schwarz's lemma. If $f(0)=0$, $|f(Z)|\leqq 1$ for $|Z|=1$, then (i) $|f(Z)|\leqq|Z|$ for all $Z\epsilon R$; (ii) $f(Z)=UZV$ for $m\neq n$, $f(Z)=UZ'V$ for $m=n$ with constant unitary matrices U, V provided $|f(Z)|=|Z|$ in a complete neighborhood of an interior point of R. Furthermore, each one-to-one analytical mapping of R is a product of displacements $f(Z)=(U_1Z+U_2)(U_3Z+U_4)^{-1}$, with

$$U'\begin{pmatrix} E_m & 0 \\ 0 & -E_n \end{pmatrix}\bar{U}=\begin{pmatrix} E_m & 0 \\ 0 & -E_n \end{pmatrix}, \quad U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

and the transposition $f(Z)=Z'$; if symmetrical matrices are considered, then the condition

$$U'\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}U=\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

has to be added. In the second paper the results are further extended to the space $A_{n,n}$ of all Z with $|Z|<1$. [See also C. L. Siegel, Amer. J. Math. 65, 1–86 (1943), in particular, pp. 8–15; these Rev. 4, 242.] *O. F. G. Schilling.*

Morita, Kiiti. Schwarz's lemma in a homogeneous space of higher dimensions. Jap. J. Math. 19, 45–56 (1944). [MF 14994]

Let L_n be the set of all skew-symmetric complex matrices Z of the space $A_{n,n}$ given by $E_n-\bar{Z}'Z>0$ or $|Z|<1$. Then each one-to-one analytic mapping of L_n is given by

$$Z\to(U_1Z+U_2)(U_3Z+U_4)^{-1},$$

where

$$\bar{U}'\begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}U=\begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix},$$
$$U'\begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}U=\begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}.$$

If $n=4$ the transformation $\psi(Z)=Z^+$ has to be added, Z^+ meaning the interchange of the element at the place (1, 4) with the element at the place (2, 3). The proof uses the Riemann metric given by

$$ds^2=\mathrm{Sp}(E-\bar{Z}'Z)^{-1}\overline{dZ}'(E-Z\bar{Z}')^{-1}dZ.$$

Furthermore, various generalized forms of Schwarz's lemma are proved, for example, $|f(Z)|\leqq|Z|$ for any $Z\varepsilon L_n$ if f is an analytic mapping with the fixed point 0. [See also C. L. Siegel, Amer. J. Math. 65, 1–86 (1943); these Rev. 4, 242.] *O. F. G. Schilling* (Chicago, Ill.).

Trjitzinsky, W. J. Problems of representation and uniqueness for functions of a complex variable. Acta Math. 78, 97–192 (1946). [MF 15601]

This is a continuation of earlier papers of the author [Ann. Sci. École Norm. Sup. (3) 55, 119–191 (1938); Acta Math. 70, 63–163 (1938)], familiarity with which is assumed. The functions considered are, for the most part, differentiable on certain sets, in general without interior points. The results established are expressed in terms of descriptive properties of the sets and classes of functions in question, and are too complicated in statement to be quoted in detail here. The author first extends his earlier work on representation of generalized monogenic functions by double integrals,

$$f(z)=h(z)-\frac{1}{2\pi i}\iint_K\frac{d\mu(e_\zeta)}{\zeta-z},$$

where K is a rectangle, $h(z)$ is analytic in K and $\mu(e)$ is an absolutely continuous additive function of Borel sets. In particular, $f(z)$ has this representation if it is "continuously monogenic" on a perfect subset F of K, that is, if it is defined and continuously differentiable on F (with respect to F). The representation is extended to certain classes of discontinuous functions.

Next, conditions are given under which certain limits of sequences of analytic functions are continuously monogenic. Conditions are found under which such limits have a kind of quasi-analytic property, so that the vanishing of the function on a set implies its vanishing on a larger set; if the class of functions considered is additive, two functions coinciding on a suitable set coincide on the larger set. The following theorem may serve as an example of the results obtained. If G is a closed bounded set, $f_n(z)=\sum_{k=1}^{n}A_{n,k}/(z-\alpha_{n,k})$, where the α's are bounded and outside G, and $|f(z)-f_n(z)|\leqq\epsilon_n$ in G, where $\epsilon_n\to0$ sufficiently rapidly, then $f(z)$ vanishes in G if $f(z)$ vanishes on a rectifiable arc in G. According to the author's earlier work, wide classes of functions have the representation in question. *R. P. Boas, Jr.*

Haskell, R. N. Areolar monogenic functions. Bull. Amer. Math. Soc. 52, 332–337 (1946). [MF 16201]

A complex function $w=u(x, y)+iv(x, y)$, of the complex variable $z=x+iy$, for which the Cauchy-Riemann equations do not necessarily hold, has been termed by Kasner a polygenic function of z. The first derivative is represented by the Kasner circle. The general second order derivative for any curvilinear path of approach has been studied by Kasner and DeCicco. The author considers the second derivative for two perpendicular rectilinear paths of approach. If this is independent of the directions of approach, w is said to be areolar monogenic and the unique value is called the Cioranescu derivative. This class of areolar monogenic functions is given by $\bar{z}A(z)+B(z)$, where A and B are analytic functions of z. The Cioranescu derivative of $w=\bar{z}A(z)+B(z)$ is $d^2w/dz^2=\bar{z}A''+B''$. Following Looman and Menchoff, the author shows how the class of areolar monogenic functions can be characterized by reducing the restrictions for the existence of the Cioranescu derivative. Finally, an analogue of Morera's theorem is obtained for this class of functions. *J. DeCicco* (Chicago, Ill.).

Differential Equations

Oppelt, W. Zum Dämpfungsgrad der Regelungsdifferentialgleichung dritter Ordnung. Arch. Elektrotechnik 37, 357–360 (1943). [MF 15627]

This note concerns a graphical representation of the solutions of the equation $z^3+A_2z^2+A_1z+1=0$, where A_1 and A_2 are positive constants. The implications of the results for the solution of the differential equation $\psi'''+A_2\psi''+A_1\psi'+\psi=0$ are discussed, with particular reference to the problem of stabilizing automatic regulators and control systems. *L. A. MacColl* (New York, N. Y.).

Nagumo, Mitio. Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. Proc. Phys.-Math. Soc. Japan (3) 24, 551–559 (1942). [MF 15034]

Consider a differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is defined over a region D in the (x, y)-plane. A subregion E of D is called "nach rechts majorant in D" if every integral curve in D, which has its initial point (x_0, y_0) in E, remains in E for $x\geqq x_0$. This notion can be extended readily to the case of a system of differential equations

$$dy_i/dx=f_i(x, y_1, y_2, \cdots, y_n),\quad i=1, 2, \cdots, n.$$

The purpose of this note is to give various necessary and sufficient conditions for a subregion to be "nach rechts majorant in D." The conditions cannot be stated in a way that is both concise and explicit. *L. A. MacColl.*

Levinson, Norman. The asymptotic behavior of a system of linear differential equations. Amer. J. Math. 68, 1–6 (1946). [MF 15483]

It is shown that, if the linear system of differential equations with constant coefficients

$$(1)\qquad y_j'(t)=\sum_{k=1}^{n}a_{jk}y_k(t),\qquad j=1, \cdots, n,$$

has all its solutions bounded as $t \to +\infty$, while the coefficients $f_{jk}(t)$ of the linear system

$$(2) \qquad x_j'(t) = \sum_{k=1}^{n} f_{jk}(t) x_k(t), \qquad j = 1, \cdots, n,$$

are continuous and satisfy $\int^\infty |f_{jk}(t) - a_{jk}| dt < \infty$, $j, k = 1, \cdots, n$, then for any solution $\{x_j(t)\}$ of (2) there is a solution $\{y_j(t)\}$ of (1) containing sinusoidal terms only, and such that $x_j(t) - y_j(t) \to 0$ as $t \to +\infty$. This theorem generalizes a result of Wintner [same J. 67, 417–430 (1945); these Rev. 7, 117]. *W. T. Reid* (Evanston, Ill.).

Weyl, Hermann. Comment on the preceding paper. Amer. J. Math. 68, 7–12 (1946). [MF 15484]

The result of the paper reviewed above is extended to a nonlinear vector differential equation

$$(1) \qquad d\mathfrak{x}/dt = A\mathfrak{x} + \mathfrak{f}(t, \mathfrak{x}(t)),$$

where A is a constant matrix and $\mathfrak{f}(t, \mathfrak{x})$ is a given vector function of t and a variable vector $\mathfrak{x}$, under the following hypotheses: (I) $\|\mathfrak{f}(t, \mathfrak{x})\| \leqq \|\mathfrak{x}\| \cdot g(t)$; (II) $\int_0^\infty g(t) dt$ converges; (III) every solution $\mathfrak{z} = \mathfrak{z}(t)$ of (2) $d\mathfrak{z}/dt = A\mathfrak{z}$ is bounded for $t \to \infty$. In addition, the author treats the inverse process of transition from a solution $\mathfrak{z}(t)$ of the linear system (2) to a solution $\mathfrak{x}(t)$ of (1); for this consideration (I) is replaced by the stronger condition

$$(\mathrm{I}^*) \qquad \mathfrak{f}(t, 0) = 0, \quad \|\mathfrak{f}(t, \mathfrak{x}) - \mathfrak{f}(t, \mathfrak{x}^*)\| \leqq \|\mathfrak{x} - \mathfrak{x}^*\| \cdot g(t),$$

while (II) and (III) are retained. *W. T. Reid.*

Bulgakov, B. V. On the method of van der Pol and its application to non-linear control problems. J. Franklin Inst. 241, 31–54 (1946). [MF 15304]

Ideas of Appleton, van der Pol, Mandelstam and Papalexi are applied to a general system of pseudo-linear differential equations

$$\sum_{k=1}^{n} f_{jk}(D) x_k = \phi_j(x_1, \cdots, x_n, t), \qquad j = 1, \cdots, n,$$

where the f_{jk} are polynomials with constant coefficients in the differential operator D and the ϕ_j are nonlinear and may also involve derivatives of x_k of not too high order. Unlike the earlier paper [same J. 235, 591–616 (1943); these Rev. 4, 245], which considered only periodic solutions, the transient case and almost periodicity are handled. Application is made to a problem in automatic control.

N. Levinson (Cambridge, Mass.).

Sato, Tunezo. On Green's functions of linear differential equations of the fourth order. Proc. Phys.-Math. Soc. Japan (3) 23, 775–783 (1941). [MF 15007]

The author discusses four examples illustrating a method for solving boundary value problems which consists of replacing the given problem by an equivalent set of two or more simpler problems. Thus, for example, the problem

$$y^{\mathrm{iv}} = f(x), \quad y(0) = y''(0) = y(1) = y''(1) = 0,$$

is equivalent to the set of two problems

$$y'' = \omega(x), \quad y(0) = y(1) = 0;$$
$$\omega'' = f(x), \quad \omega(0) = \omega(1) = 0.$$

Consequently, we have

$$y(x) = \int_0^1 f(\xi) d\xi \int_0^1 G(x, \eta) G(\eta, \xi) d\eta,$$

where

$$G(x, \xi) = \begin{cases} x(1-\xi), & x < \xi, \\ \xi(1-x), & x > \xi. \end{cases}$$

The method also affords an easy determination of eigenvalues and eigenfunctions. *L. A. MacColl.*

Borg, Göran. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. Acta Math. 78, 1–96 (1946). [MF 15600]

The usual Sturm-Liouville problem consists in determining the spectrum for a differential equation $y'' + (\lambda + \varphi(x)) = 0$ when the boundary conditions are given. The present paper studies the inverse problem of recovering $\varphi(x)$ when the spectrum and the boundary conditions which give rise to it are known. For example, the problem of discovering the equations of a string from a knowledge of the fundamentals and overtones is of the inverse kind. The conclusions are as follows. (i) The solution of the inverse problem is not unique in general. (ii) The solution is unique if it is prescribed that $\varphi(x) = \varphi(\pi - x)$, provided that the boundary condition has one of the forms $y(0) = y(\pi) = 0$ or $y'(0) = y'(\pi) = 0$. (iii) To an arbitrary Sturm-Liouville problem corresponds an associated one with the following property: if the spectrum of the associated problem is known in addition to the prescribed one, then $\varphi(x)$ is uniquely determined. The chief tools used in the proofs are the theory of Fourier series and von Koch's theory of infinite determinants. The paper concludes with a study of physical applications; in particular, it is shown that under appropriate conditions the string problem mentioned above has a unique solution.

H. Pollard (New Haven, Conn.).

Nagumo, Mitio. Über das Anfangswertproblem partieller Differentialgleichungen. Jap. J. Math. 18, 41–47 (1942). [MF 14966]

The author treats the existence and uniqueness question for the initial value problem for the system of partial differential equations

$$(1) \quad \partial u_\mu/\partial t = F^\mu(t, x_1, \cdots, x_k, u_1, \cdots, u_m, p_{11}, \cdots, p_{mk}),$$
$$p_{ij} = \partial u_i/\partial x_j; \quad \mu = 1, \cdots, m,$$

where the F^μ are continuous in (t, x, u, p) and analytic in (x, u, p). By differentiation with respect to the x_i, (1) is reduced to an equivalent quasilinear system, which in turn is shown to be equivalent to a system of functional equations

$$q_\mu(t, x) = \phi_\mu\{q_1(t, x), \cdots, q_m(t, x)\}, \quad \mu = 1, \cdots, m,$$

where the ϕ_μ are functionals which (in the small) turn out to satisfy the hypotheses of Schauder's fixed-point theorem. This proves the existence in the small of a solution of the given initial value problem. The uniqueness proof is given by a method similar to one used in the theory of ordinary differential equations, first in the small and then extended to the whole domain in which the solution exists.

E. H. Rothe (Ann Arbor, Mich.).

Walsh, J. L. Note on the location of the critical points of harmonic functions. Bull. Amer. Math. Soc. 52, 346–347 (1946). [MF 16203]

The following theorem is stated together with a brief indication of a method of proof. In the extended (x, y)-plane let R_0 be a simply-connected region bounded by a continuum C_0 not a single point and let the disjoint continua $C_1, \cdots, C_n$ lie interior to R_0 and together with C_0 bound a subregion R of R_0. By means of a conformal map of R_0 onto the unit circle we define in R_0 non-Euclidean lines, the images of arbitrary circles orthogonal to the unit circle.

Denote by Π the smallest closed non-Euclidean convex region in R_0 which contains $C_1, \cdots, C_n$. Let the function $u(x, y)$ be harmonic interior to R, continuous in the closure of R, with the values zero on C_0 and unity on $C_1, \cdots, C_n$. Then the critical points of $u(x, y)$ in R are $n-1$ in number and lie in Π. An extension is also announced.

M. H. Heins (Cambridge, Mass.).

Cinquini-Cibrario, Maria. Sopra alcune questioni relative ad equazioni ellittico-paraboliche del secondo tipo misto. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 365–383 (1942). [MF 16254]

Let D_1 (D_2) be a finite region of the half plane $x>0$ ($x<0$), part of whose boundary is a segment $M\leqq y\leqq N$ of the y-axis, and let γ_1 (γ_2) be the boundary of D_1 (D_2) except for $M<y<N$ of the y-axis segment. The author shows that, if continuous values are assigned on the curve γ_1, there exists in D_1 a unique solution for each equation of the type

$$(*)\qquad k^{-2}x^{2(k+1)}u_{xx}+u_{yy}=0, \qquad k=1, 2, 3, \cdots,$$

that takes on these boundary values. Under certain continuity conditions on γ_1 and the assigned boundary values, such solutions of (*) are shown to possess continuous first and second partial derivatives on the segment $M<y<N$ of the boundary of D_1. If continuous values are assigned to the curve $\gamma_1+\gamma_2$, it is proved that there exists a unique solution of (*) in the region $D_1+D_2+(x=0, M<y<N)$ having these boundary values; and furthermore, under suitable continuity conditions, that such solutions are analytic in D_1 and in D_2 (the function in D_1 may not be an analytic continuation of the function in D_2). Similar results are shown to hold for the equation $x^2u_{xx}+u_{yy}=0$. The present paper extends earlier results of the author [Ann. Mat. Pura Appl. (4) 14, 215–247 (1936); Rend. Circ. Mat. Palermo 49, 347–372 (1935); Atti Accad. Sci. Torino Cl. Fis. Mat. Nat. 76, 105–124 (1941); these Rev. 2, 365]. *F. G. Dressel.*

Schaaf, Samuel A. A cylinder cooling problem. Quart. Appl. Math. 3, 356–360 (1946). [MF 14526]

A solid infinite cylinder $r<a$ of one material is surrounded by a solid medium $r>a$ of another material. The formula for the temperatures $T(r, t)$ is derived when the cylinder has one uniform temperature initially and the surrounding medium another. The formula involves infinite integrals of combinations of Bessel functions arrived at by reducing certain inverse Laplace transforms to real integrals.

R. V. Churchill.

Fowler, Clarence M. Analysis of numerical solutions of transient heat-flow problems. Quart. Appl. Math. 3, 361–376 (1946). [MF 14527]

Solutions of the difference equation for the temperatures $T_{x,t}$ under one-dimensional transfer of heat by conduction are considered. The examples include both finite and infinite slabs with some variety of simple end conditions, all with uniform initial temperatures. The author employs contour integrals of particular solutions of the difference equation to give the temperatures in terms of polynomials and trigonometric functions. The convergence of his solutions to the well-known solutions of the corresponding problems in partial differential equations, as the differences Δx and Δt tend to zero, is investigated. Conditions for the convergence to the steady-state solution, as t tends to infinity, are also noted. *R. V. Churchill* (Ann Arbor, Mich.).

Charny, I. A. On a modification of Forchheimer's problem. C. R. (Doklady) Acad. Sci. URSS (N.S.) 48, 27–30 (1945). [MF 15227]

A long cylindrical tube of given outer radius is buried at a given depth below the earth's surface. The outer surface of the tube is maintained at a uniform temperature and is in perfect contact with the soil. Heat transfer takes place at the earth's surface according to the linear law that the flux of heat into the atmosphere is proportional to the temperature of the earth's surface. The author solves the problem of determining the steady-state temperatures in the soil and the time-rate of loss of heat from a unit length of the tube. He uses a conformal transformation to change the problem into one for a circular ring which he solves with the aid of a series in the complex plane. A simple approximate solution is included and results are compared briefly with experimental information. *R. V. Churchill.*

George, Jean-Claude. Résolution de l'équation de la diffusion par une méthode symbolique à deux opérateurs. C. R. Acad. Sci. Paris 219, 405–407 (1944). [MF 15274]

The Laplace transform is used to solve the problem of one-dimensional diffusion in the case in which the initial concentration is $c=c_1$ when $0<x<h_1$, $c=c_2$ when $h_1<x<h_2$ and $c=c_3$ when $h_2<x<h$. The boundaries $x=0$ and $x=h$ are assumed impervious to the diffusing substance. Formulas for $c(x, t)$ are found in the two well-known forms: in terms of error functions and in terms of trigonometric and exponential functions. *R. V. Churchill* (Ann Arbor, Mich.).

Sakadi, Zyuro. On cooling of a semi-infinite rod by current of cold fluid. Proc. Phys.-Math. Soc. Japan (3) 24, 715–718 (1942). [MF 15040]

A rod $r\leqq r_0$, $x\geqq 0$ along the positive half of the x-axis has a circular cross section that is small enough so that the average steady temperature $T_2(x)$ over the cross section can be used instead of the actual steady temperature. A current of cold fluid $r>r_0$ is circulated along the rod with a constant velocity parallel to the rod. The temperature of the fluid $T_1(x, r)$ at the plane $x=0$ is $T_1(0, r)=0$, and $T_2(0)=C$. A boundary value problem for the steady-state temperatures T_1 and T_2 ($x\geqq 0$, $r\geqq 0$), assuming heat transfer by conduction in both media, is set up. A method of solution which involves the solving of an infinite integral equation is pointed out. *R. V. Churchill* (Ann Arbor, Mich.).

Sakadi, Zyuro. On thermal stress in an elastic solid body. Proc. Phys.-Math. Soc. Japan (3) 25, 673–685 (1943). [MF 15079]

The first order approximation for thermal stresses in a homogeneous isotropic elastic solid is

$$(1)\qquad \partial^2\xi_i/\partial t^2=(\lambda+\mu)\partial\sigma/\partial x^i+\mu\nabla^2\xi_i-\nu\partial T/\partial x^i,$$

$$(2)\qquad \partial T/\partial t=a^2\nabla^2T,$$

where $\{\xi_i\}$ is the displacement vector, σ the dilatation, T the temperature and the other symbols are conventional. The interest is in the case where $\partial^2\xi_i/\partial t^2$ is not negligible. The author treats several instances under conditions of symmetry, for example, the one-dimensional problem for the half space $x^1>0$ and the finite and infinite sphere with radial variation alone. The method consists of taking a standard solution for the heat equation (2) and introducing this in (1). For instance, the periodic solution $\Re \exp(ia^2\omega t-(i\omega)^{\frac{1}{2}}x)$ and the point source solution are

used in the appropriate problems. Under the symmetry conditions imposed, integral representations for ξ are obtained in a straightforward manner. *D. G. Bourgin.*

Jaeger, J. C. On thermal stresses in circular cylinders. Philos. Mag. (7) 36, 418–428 (1945). [MF 15479]

Radial stresses σ_r and tangential stresses σ_θ are computed here, first in the infinitely long solid cylinder $r \leqq a$ initially at a uniform temperature and with its surface $r=a$ kept at temperature zero thereafter. Corresponding computations are made for hollow cylinders $a \leqq r \leqq b$, with the surfaces $r=a$ and $r=b$ kept at uniform temperatures v_1 and v_2. The problem is generalized to allow for radiation, under the linear law of heat transfer, at the surface of the solid cylinder or at either surface of the hollow cylinder. An insulated surface is a special case. The author presents tables and graphs in dimensionless units which give the numerical values of the stresses in these cylinders at various points and times. These tabulations of numerical results, which are of considerable practical importance, form the principal objective of the paper. The stresses are computed by combining known formulas for the stresses in terms of temperatures with formulas for the temperatures. The author takes advantage of asymptotic formulas and operational methods in making his calculations.

R. V. Churchill (Ann Arbor, Mich.).

Rosenblatt, Alfred. On the hyperbolic horn with elliptic section. I. Fundamental equations. Expression for the impedance. Revista Ci., Lima 47, 301–324, 361–388 (1945). (Spanish) [MF 14484]

Considérant l'équation $c^2\Delta V - \partial^2 V/\partial t^2 = 0$ d'une onde qui se propage à l'intérieur d'un corne dont la paroi est représentée par un hyperboloide hyperbolique, l'auteur prend comme système de référence le système triple orthogonal auquel appartient la paroi. La solution générale est alors développée en série d'une suite de solutions fondamentales représentées par des produits de trois fonctions de Lamé, respectivement des trois coordonnées curvilignes.

B. Levi (Rosario).

Fremberg, Nils Erik. Proof of a theorem of M. Riesz concerning a generalization of the Riemann-Liouville integral. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 15, no. 27, 265–276 (1945). [MF 15521]

M. Riesz [C. R. Congrès Int. Math., Oslo, 1936, vol. 2, pp. 44–45] generalized the Riemann-Liouville integral of fractional order to a space of metric $ds^2 = dt^2 - \sum_{\nu=1}^{m-1} dx_\nu^2$. If r is the distance from a fixed point P to a variable point Q, the Riesz integral is

$$I^\alpha f(P) = (1/H_m(\alpha)) \int_D f(Q) r^{\alpha-m} dQ,$$

where $H_m(\alpha)$ is a certain constant and D is the region bounded by the retrograde cone $r=0$ and a surface S; $I^\alpha f(P)$ is an analytic function of the complex variable α regular in $\Re(\alpha) > m-2$.

This integral can be used to solve the problem of Cauchy for the generalized wave equation with data on S, provided that it is possible to continue analytically the function $I^\alpha f(P)$ up to $\alpha = 0$. The present paper gives the first proof of the possibility of this analytical continuation for general values of m and for a general surface S, assuming that S and f satisfy certain conditions of continuity and differentiability. It is not possible to give a brief account of the method here.

E. T. Copson (Dundee).

Difference Equations, Special Functional Equations

Lublin, Mogens. On a class of nonlinear difference equations. Mat. Tidsskr. B. 1946, 120–128 (1946). (Danish) [MF 16313]

The author studies the nonlinear difference equation

$$(1) \qquad y(x+1) = \sum_{k=0}^{n} a_k y^k(x)$$

in the complex plane. It is shown that any solution which satisfies the two inequalities

$$(2) \qquad |y(x)| > 1, \quad |a_n y(x)| > 1 + |a_0| + \cdots + |a_{n-1}|$$

in the strip $x_0 \leqq \Re(x) < x_0 + 1$ is of the form

$$y(x) = \varphi(\exp(\pi(x) n^x)),$$

where $\pi(x)$ is periodic with period 1 and $\varphi(y)$ is a function analytic in the domain defined by (2), has a simple pole at infinity, and satisfies the functional equation $\varphi(z^n) = \sum_{k=0}^{n} a_k \varphi^k(z)$.

W. Feller (Ithaca, N. Y.).

Sievert, Rolf M. Zur theoretisch-mathematischen Behandlung des Problems der biologischen Strahlenwirkung. Acta Radiologica 22, 237–251 (1941). [MF 16629]

From general considerations a difference-differential equation is obtained whose solution, obtained by graphical methods, is compared with observations.

W. Feller (Ithaca, N. Y.).

van der Werff, J. Th. Die mathematische Theorie der biologischen Reaktionserscheinungen, besonders nach Röntgenbestrahlung. Acta Radiologica 23, 603–621 (1942). [MF 16541]

Instead of the difference-differential equation of the preceding review, the author obtains a differential equation. [The latter could more conveniently be rewritten as an integral equation of the renewal type].

W. Feller (Ithaca, N. Y.).

van der Werff, J. Th. On some functional equations arising in a theory of the biological phenomena of reaction to X-rays. Nieuw Arch. Wiskunde (2) 21, 197–211 (1943). (Dutch) [MF 15704]

Let $x(t)$ be a solution of the difference-differential equations

$$x'(t) = -ax(t) + b\{x_0 - x(t-\tau)\}, \qquad 0 \leqq t \leqq T,$$
$$x'(t) = b\{x_0 - x(t-\tau)\}, \qquad t > T;$$

here a, b, x_0 and τ are positive constants, $0 \leqq T \leqq \infty$, and it is assumed that $x(t) = x_0$ for $t \leqq 0$. The author establishes conditions under which $x(t)$ will tend to a constant as $t \to \infty$. In particular, when $T = \infty$ it turns out that $x(t) \to$ constant if and only if either $b < a$ or $a < b$ and $b\tau < \varphi \sin \varphi$, where φ is the smallest positive angle for which $a + b \cos \varphi = 0$. The proof is based on the study of the generating function $\sum A_n(\psi) s^n$, where $A_n(\psi) = x(n+\psi)$, $0 < \psi < 1$. [For the biological theory mentioned in the title cf. the two preceding reviews.]

W. Feller (Ithaca, N. Y.).

Bădescu, Radu. Sur l'équation fonctionnelle de Poinsot généralisée. Bull. Math. Phys. Éc. Polytech. Bucarest 11, 3–29 (1940). [MF 13545]

The author considers the functional equation

$$(1)\qquad f(z)-\lambda f(\alpha z)=p(z)$$

for complex z, α and λ with $|\alpha|\neq 1$. He begins with the case in which $p(z)$ is regular near $z=0$, showing that the equation has a unique solution $f(z)$ regular near $z=0$ if λ takes none of the values α^{-n} ($n=0, 1, 2, \cdots$). The homogeneous equation (with $p(z)\equiv 0$) has solutions of this type if and only if λ takes one of these values, and in this case (1) has a regular solution if and only if $p_n=0$, where p_n is the coefficient of z^n in the Taylor expansion of $p(z)$. He also shows that, when $p_n\neq 0$, there exist solutions differing from a regular function by a multiple of $z^n \log z$.

The author then investigates the solutions of (1) when $p(z)$ has an essential singularity at $z=0$, the solution being now permitted to have an essential singularity. In this case the homogeneous equation has solutions of a new type depending on an arbitrary function and expansible in a Laurent series in a certain annulus. The analytic behaviour of all the solutions found is examined in some detail, and a number of particular examples are considered.

F. Smithies (Cambridge, England).

van der Corput, J. G. A remarkable family. Euclides 18, 50–78 (1941). [MF 15688]

The author discusses a number of functional equations suggested by trigonometric identities. The following will serve as examples.

$$(1)\qquad f(x)=2f(\tfrac{1}{2}x)/\{1-f(\tfrac{1}{2}x)^2\}.$$

If $b>0$, $\frac{1}{2}b<a<b$, and $F(x)$, never equal to 0 or $\pm i$, is of class C^∞ in $a\leqq x\leqq b$, then there is a solution of (1), of class C^∞ in $0<x\leqq b$, coinciding with $F(x)$ on $a\leqq x\leqq b$. However, the only solutions of (1) which are differentiable and vanish at $x=0$ are the functions $\tan kx$; the only solutions which are continuous at $x=0$ and have $f(0)=\pm i$ are the constants $\pm i$.

$$(2)\qquad f(x)=2f(\tfrac{1}{2}x)^2-1.$$

Solutions of (2) are not determined by being differentiable at $x=0$ and having $f(0)=1$ but the only solutions which are twice differentiable at $x=0$ and have $f(0)=1$ are the functions $\cos kx$ (where k is complex). *R. P. Boas, Jr.*

Ridder, J. On the additive functional equation and an additive functional congruence. Euclides 18, 84–92 (1941). [MF 15689]

A discussion in the terminology of abstract spaces of the functional equations

$$f(x+y)=f(x)+f(y),\quad f(x+y)\equiv f(x)+f(y)\pmod{2\pi},$$
$$f(x-y)=f(x)f(y)+g(x)g(y).$$

R. P. Boas, Jr. (Providence, R. I.).

Haruki, Hiroshi. On characterisation of the elemental functions. Proc. Phys.-Math. Soc. Japan (3) 25, 457–459 (1943). [MF 15064]

Five theorems giving characteristic properties of some elementary functions. Theorem V, in the reviewer's formulation, is as follows. If a single-valued complex function $f(x+iy)$ is regular at the origin, if $f(0)=f'(0)=1$, and if its "Betragsfläche" (that is, the surface representing $|f|^2$ as a function of x and y) has only parabolic points, then $f(z)=e^z$.

F. John (New York, N. Y.).

Haruki, Hiroshi. On a certain simultaneous functional equation concerning the elliptic functions. Proc. Phys.-Math. Soc. Japan (3) 24, 450–454 (1942). [MF 15032]

The author shows, under general conditions of continuity, that the only solutions of the three functional equations

$$f(x+y)=\frac{f(x)g(y)h(y)+f(y)g(x)h(x)}{1-k^2f^2(x)f^2(y)},$$
$$g(x+y)=\frac{g(x)g(y)-f(x)f(y)h(x)h(y)}{1-k^2f^2(x)f^2(y)},$$
$$h(x+y)=\frac{h(x)h(y)-k^2f(x)f(y)g(x)g(y)}{1-k^2f^2(x)f^2(y)},$$

$-\infty<x<+\infty$, $-\infty<y<+\infty$, are

$$\text{(A)}\quad\begin{cases}(1)\ \ f(x)=g(x)=h(x)=0,\\ (2)\ \ f(x)=a,\quad g(x)=b,\quad h(x)=c\end{cases}$$

(with certain algebraic relations among a, b, c and k);

$$\text{(B)}\quad\begin{cases}(1)\ \ f(x)=h(x)=0;\quad g(x)=e^{\beta x},\\ (2)\ \ f(x)=g(x)=0;\quad h(x)=e^{\gamma x},\\ (3)\ \ f(x)=0,\quad g(x)=e^{\beta x},\quad h(x)=e^{\gamma x}\end{cases}$$

(β and γ are arbitrary real numbers);

$$\text{(C)}\qquad f(x)=\operatorname{sn}\alpha x,\quad g(x)=\operatorname{cn}\alpha x,\quad h(x)=\operatorname{dn}\alpha x$$

(α is an arbitrary real number). The main point of the demonstration consists in the fact that the differentiability of $f(x)$, $g(x)$ and $h(x)$ in $-\infty<x<+\infty$ is not supposed, but is demonstrated as a consequence of the functional equations and the continuity of $f(x)$, $g(x)$ and $h(x)$.

S. C. van Veen (Delft).

Calculus of Variations

Reid, William T. A note on the Du Bois-Reymond equations in the calculus of variations. Bull. Amer. Math. Soc. 52, 158–166 (1946). [MF 15456]

The author gives a direct proof that functions $y_1(x), \cdots, y_n(x)$ minimizing an integral

$$I(y)=\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

satisfy, almost everywhere, the equations

$$f_{y_i'}-\int_{x_1}^{x} f_{y_i}dx=\text{constant},\qquad f-y_i'f_{y_i'}-\int_{x_1}^{x} f_x dx=\text{constant}.$$

The hypotheses are believed to be weaker than any previously published; in particular, the minimizing functions $y_i(x)$ are not even required to be of bounded variation. A form of the fundamental lemma of the calculus of variations is noted as a special case. *L. M. Graves.*

Belgodère, Paul. Courbure moyenne généralisée. C. R. Acad. Sci. Paris 218, 739–740 (1944). [MF 15325]

The generalized mean curvature of a surface $z=Z(x, y)$ is defined as $\frac{1}{2}(Cr-2Bs+At)/(AC-B^2)$ when the first fundamental form is $ds^2=Adx^2+2Bdxdy+Cdy^2$. The author discusses the connection between the family of generalized mean curvatures and partial differential equations of the form $Cr-2Bs+At=f$, and points out that the equation for the extremals of the integral $\iint g(x, y, z, p, q)dxdy$ is of this type. Extensions to spaces of n dimensions are indicated. The proofs are reserved for a later publication.

J. E. Wilkins, Jr. (Chicago, Ill.).

Belgodère, Paul. Extrémales d'une intégrale de surface $\iint g(p, q)dxdy$. C. R. Acad. Sci. Paris **219**, 272–273 (1944). [MF 15256]

The author states that the properties of the envelope of the plane $z=px+qy-g(p, q)$ characterize the extremal surfaces of the integral $\iint g(p, q)dxdy$. For example, if the envelope is a curve the extremal surfaces are ruled surfaces whose generators are parallel to the tangents to the curve. The proof of these results is promised in a later publication.

J. E. Wilkins, Jr. (Chicago, Ill.).

Pâquet, P.-V. Sur la réduction de la variation complète d'une intégrale n-uple à celle d'une intégrale $(n-1)$-uple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **26**, 314–326 (1940). [MF 13834]

The author obtains two formulations of the first variation for multiple integral problems of the calculus of variations by means of Green's lemma. The work is formal in character and is based on previous results of De Donder.

H. H. Goldstine (Princeton, N. J.).

De Donder, Th. Sur les problèmes bien posés par le calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 293–301 (1943). [MF 13847]

The author considers the first-order conditions for multiple integral problems of the calculus of variations. These conditions include transversality conditions in the case of variable boundaries. In the case where the integrand is of the form $f(x^i, y^j, y^j_k)$, with $y^j_k=\partial y^j/\partial x_k$, his side conditions are of the form $\varphi_r(x^i, y^j)=0$. Proofs are not given in this note but are promised for a subsequent work.

H. H. Goldstine (Princeton, N. J.).

Lepage, Th. H. J. Champs stationnaires, champs géodésiques et formes intégrables. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 73–92, **Erratum**, 885 (1942). [MF 13650]

Lepage, Th. Champs stationnaires, champs géodésiques et formes intégrables. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 247–265 (1942). [MF 13661]

Let $L(t_1, \cdots, t_u; x_1, \cdots, x_n; p_{11}, \cdots, p_{nu})$ be a function holomorphic in its arguments and let Ω be the set of symbolic forms satisfying

$$(1)\quad \Omega\equiv L dt_1 \cdots dt_u, \quad d\Omega\equiv 0(\text{mod } \omega_i), \quad \omega_i=dx_i-p_{i\alpha}dt_\alpha.$$

If we set $L_{i\alpha}=\partial L/\partial p_{i\alpha}$ and

$$(\alpha-1, \omega_i, \alpha+1)\equiv dt_1 \cdots dt_{\alpha-1}\omega_i dt_{\alpha+1} \cdots dt_u,$$

then

$$\Omega_1=Ldt_1 \cdots dt_u+L_{i\alpha}(\alpha-1, \omega_i, \alpha+1),$$

$$\Omega_2=\Omega_1+\lambda_{i\alpha;j\beta}(\alpha-1, \omega_i, \alpha+1; \beta-1, \omega_j, \beta+1),$$

where $\lambda_{i\alpha;j\beta}$ are arbitrary functions,

$$\Omega^*=L^{1-u}\prod_{\alpha=1}^{u}(Ldt_\alpha+L_{i\alpha}\omega_i)$$

are three forms satisfying the congruence (1). The present papers are a continuation of an earlier paper [same Bull. (5) **27**, 27–46 (1941); these Rev. **4**, 143]. The object of the author is to establish a theory similar to the calculus of variations for the forms satisfying (1). The first paper is mainly devoted to definitions of the following type. Let E_u be a surface given by $x_i=x_i(t_1, \cdots, t_u)$, where t_i belongs to a region G. A set of functions $p_{i\alpha}=p_{i\alpha}(x, t)$ is said to be a field enveloping E_u if, for all $t\epsilon G$, $p_{i\alpha}(x(t), t)=\partial x_i/\partial t_\alpha$. A field $(p_{i\alpha})$ enveloping E_u is said to be stationary if it renders any member of the set of forms Ω integrable. Stationary fields S that render Ω_1, Ω_2 or Ω^* integrable are denoted by S_1, S_2, S^*. The form Ω^* is the unique form of minimum rank of the set Ω and its stationary fields S^* are called geodesic fields. Let $I(E_u)$ be the integral of Ω over the surface E_u. Then if $(p_{i\alpha})$ envelops E_u, $I(E_u)=\int_G L(t, x, \partial x/\partial t)dt_1 \cdots dt_u$, since $\omega_i=0$ on E_u. If $\bar{E}_u$ is another surface $\bar{x}_i=\bar{x}_i(t)$, $t\epsilon\bar{G}$, the first variation ΔI is defined as $I(\bar{E})-I(E)$. The author shows that, if the field $(p_{i\alpha})$ envelops E_u,

$$(2)\quad \Delta I=\int_{\bar{G}} L(t, \bar{x}, \partial\bar{x}/\partial t)dt-\int_G L(t, x, \partial x/\partial t)dt+\int_{\bar{E}_u} E\, dt,$$

$dt\equiv dt_1 \cdots dt_u$, where E is a function that vanishes with the difference $p_{i\alpha}(\bar{x}, t)-\partial\bar{x}_i/\partial t_\alpha$ and varies from form to form of the set of forms Ω. The first two integrals on the right of (2) are called the principal part of ΔI. In a stationary field S the principal part of ΔI vanishes if E_u and $\bar{E}_u$ have the same boundaries (in this case, the function E in the last integral of (2) may change form, but it still vanishes with $p_{i\alpha}-\partial\bar{x}_i/\partial t_\alpha$). The author also proves that the principal part of ΔI vanishes, even if the boundaries of E and $\bar{E}$ are not the same, when $\bar{E}$ is a section of a family of curves passing through E known as the characteristic curves of the integrable form Ω in the field S. In the case of a family of surfaces $\bar{E}_u$ depending on a parameter τ and reducing to E_u for $\tau=0$, the author sets up a definition (too long to give here) of the first variation δI of the integral $I(E_u)$.

In the second paper the author considers the problem of finding the $p_{i\alpha}$ so that one of the forms Ω is integrable ($d\Omega=0$). In particular, methods for constructing the stationary fields S_1, S_2, S^* are given.

[The right side of (2.8), page 76, defines the negative of E_1. Since the $\pi_{i\alpha}$ are to be considered functions of x_i, formula (13.7'), page 249, should read $dH/dx_i=\partial\pi_{i\alpha}/\partial t_\alpha$.]

F. G. Dressel (Durham, N. C.).

GEOMETRY

Jessen, Børge. On equivalence of aggregates of regular polyhedra. Mat. Tidsskr. B. **1946**, 145–148 (1946). (Danish) [MF 16316]

Lebesgue [Ann. Soc. Polonaise Math. **17**, 193–226 (1939)] gave a necessary condition that two aggregates of regular polyhedra be equivalent, that is, may be divided into polyhedra $P_1, \cdots, P_n$ and $P_1', \cdots, P_n'$, respectively, such that P_i is congruent to P_i'. Using a result of Sydler [Comment. Math. Helv. **16**, 266–273 (1944); these Rev. **6**, 183] the author shows that for aggregates having the same volume Lebesgue's condition is also sufficient.

W. Feller.

Hjelmslev, Johannes. Beiträge zur nicht-Eudoxischen Geometrie. I, II. Danske Vid. Selsk. Math.-Fys. Medd. **21**, no. 5, 26 pp. (1944). [MF 15403]

This paper is based on Hilbert's "Grundlagen der Geometrie" and assumes his sets of axioms I–III [cf. the 2d–7th editions]. Two lengths (angles) are called equivalent if a suitable finite multiple of the smaller length (angle) is larger than the larger one. If a and b are not equivalent and if $a<b$, then a is called inferior to b. The Eudoxian (Archimedean) axiom states that any two lengths are equivalent. This paper deals mainly with two aspects of non-Archimedean geometry, (1) equivalence, (2) area.

(1) If two angles of a triangle are equivalent, then the opposite sides are so too. Thus the existence of nonequivalent lengths implies that of nonequivalent angles. Another corollary is that to any length (angle) a smaller one exists. Other theorems on the intersection of straight lines and on the equivalence of sides or angles of a triangle or a convex polygon hold only in the Euclidean and elliptic cases.

(2) In the Euclidean case, the area of a polygon can be defined if the unit of length is equivalent to its largest side. Two polygons are equal in area (zerlegungsgleich) if they can be decomposed into congruent triangles. The theorem that two triangles with the same area are equal in area does not remain valid in non-Archimedean geometry. The author shows, however, that two triangles (more generally, two convex polygons) with the same area are equal in area if their largest sides are equivalent. Similar theorems are proved in the non-Euclidean cases. *P. Scherk.*

Sicardi, Francesco. Su di una particolare metrica non archimedea nei fasci di rette. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 452–467 (1941). [MF 16264]

Consider a generalized number (ϵ, α), where $\epsilon=\pm 1$, α is a real number, and addition is defined by $(\epsilon, \alpha)+(\epsilon', \alpha') =(\epsilon\epsilon', \alpha+\alpha')$. Let a cyclic order be imparted as follows: $(1, \alpha)$ with α increasing, then $(1, +\infty)=\omega_1=(-1, +\infty)$, $(-1, \alpha)$ with α decreasing, $(-1, -\infty)=\omega_2=(1, -\infty)$. Since $n(\epsilon, \alpha)=(\epsilon^n, n\alpha)$ for any integer n, no multiple of a number $(1, \alpha)$ "of the first class" can exceed a number $(-1, \alpha')$ "of the second class;" thus the system is non-Archimedean. This algebra is applied to a modified Minkowskian geometry due to Bencivenga, wherein $ds^2=|dx^2-dy^2|$. Let i and i_1 denote the isotropic lines $y=\pm x$, while r and s are two other lines through the origin. A suitable definition for the angle from r to s is found to be $(\operatorname{sgn}\beta, k\log|\beta|)$, where β is the cross ratio (i, i_1, r, s).

The reviewer has slightly modified the definition of the new number system, because the author's efforts to make the (ϵ, α) additive for ϵ as well as for α led to systematic errors which could not be corrected without making the formulas very unwieldy. *H. S. M. Coxeter.*

Coxeter, H. S. M. Quaternions and reflections. Amer. Math. Monthly 53, 136–146 (1946). [MF 15755]

By considering a rotation in space to be a product of two reflections, the author obtains a simple and natural derivation of Cayley's expression $x\rightarrow ax\bar{a}$ for a rotation through an angle φ about the line through the origin with direction cosines p_1, p_2, p_3, x being a pure quaternion representing a point and a being the unit quaternion

$$a=\cos\tfrac{1}{2}\varphi+(p_1i+p_2j+p_3k)\sin\tfrac{1}{2}\varphi.$$

The author then considers a quaternion to represent a point in Cartesian 4-space and derives similar formulae for reflections and rotations in this space. In particular, every orthogonal transformation in four dimensions is either $x\rightarrow axb$ or $x\rightarrow a\bar{x}b$. Application is made to elliptic geometry. *C. C. MacDuffee* (Madison, Wis.).

Baer, Reinhold. Null systems in projective space. Bull. Amer. Math. Soc. 51, 903–906 (1945). [MF 14457]

The fact that the existence of a null-polarity implies that the dimension $n>1$ of the projective space is odd was proved by Brauer on the assumption that the underlying field is commutative [Bull. Amer. Math. Soc. 42, 247–254 (1936)]. Here commutativity is shown to follow from the existence of the correspondence. *G. de B. Robinson.*

Baer, Reinhold. Polarities in finite projective planes. Bull. Amer. Math. Soc. 52, 77–93 (1946). [MF 15445]

The notion of a polarity in projective geometry is fundamental and has many significant applications. In this paper the theory is developed in the finite projective plane without assuming Desargues' theorem. The arguments are brief; we quote a few of the conclusions. If the number of points on a line is $n+1$, then it is shown that $M\geqq n+1$ points lie on their own polar lines (these are called absolute points). For n even, $M=n+1$ if and only if all the absolute points are collinear; for n odd, if and only if not more than two absolute points lie on a given line. Otherwise $M>n+1$, n is a square and every prime divisor of n is a divisor of $M-1$.

If a line h does not contain its own pole, that is, is not absolute, but does contain absolute points, it is said to be elliptic; dually, a point H is hyperbolic if H is not absolute, but absolute lines do contain H. The polarity under consideration is said to be regular if any two elliptic lines contain the same number $i+1$ of absolute points. The total number of absolute points is $in+1$ and it is shown that the number of hyperbolic points is $(in+1)n(i+1)^{-1}$. If Desargues' theorem holds and p is odd, a polarity is always regular with $i=1$ or $i^2=n$; if $p=2$, either all the absolute points are collinear or the polarity is regular with $i^2=n$. *G. de B. Robinson* (Toronto, Ont.).

Baer, Reinhold. Projectivities with fixed points on every line of the plane. Bull. Amer. Math. Soc. 52, 273–286 (1946). [MF 16191]

A projectivity (sometimes called a collineation) of a plane is a 1 to 1 and exhaustive mapping of the points of a plane onto themselves and of the lines onto themselves which preserves incidences. The author remarks that one may expect the structure of a projectivity to be dominated by the structure of the system of its fixed elements, provided this system is not "too small." This paper successfully characterizes a class of projectivities, called quasi-perspectivities, which have a fairly large number of fixed elements. Precisely, a quasi-perspectivity is defined as a projectivity in which every line carries a fixed point, and, as is proved to be equivalent, in which every point lies on a fixed line. The principal results are as follows. In a Desarguesian plane over the skew field G a quasi-perspectivity is either an involution or is determined by an automorphism of G given by $x^{\alpha}=(1+c)x(1+c)^{-1}$ for every x of G, where $c^{-1}+c$ but not c is in the center of G. In a finite plane with $n+1$ points on a line any projectivity with as many as n fixed points is a quasi-perspectivity. Either this quasi-perspectivity is a perspectivity or $n=i^2$ and the fixed elements form a subplane with $i+1$ points on a line. *M. Hall.*

Fabricius-Bjerre, Fr. Über geschlossene Kurven $(n+1)$-ter Ordnung im $\mathfrak{R}^n$ mit einer Anwendung auf ebene Kurven der konischen Ordnung 5 und 6. Danske Vid. Selsk. Math.-Fys. Medd. 20, no. 1, 25 pp. (1942). [MF 15399]

The author studies closed differentiable curves C_{n+1} of linear order $n+1$ in real projective n-space $\mathfrak{R}^n$, that is, curves whose maximum number of points of intersection with any $\mathfrak{R}^{n-1}$ is equal to $n+1$. He proves the theorem that the sum of the multiplicities of the singular points of the C_{n+1} is at most $n+1$ and characterizes those curves C'_{n+1} of order $n+1$ in $\mathfrak{R}^n$ for which the equality sign holds; they are identical with those C_{n+1} that have at most $n-1$ points in common with any $\mathfrak{R}^{n-2}$. These results coincide with those of Linsman [Bull. Soc. Roy. Sci. Liége 10, 350–354 (1941); these Rev. 7, 70] and of the reviewer [Proc. Nat. Acad. Sci.

U. S. A. 27, 181–182 (1941); these Rev. 2, 299]. The methods are similar.

A C_{n+1}' is decomposed by its singular points into arcs of order n. By projecting a closed curve of order $n+1$ in $\mathfrak{R}^{n+1}$ from suitable points, the existence of all the possible types of the C_{n+1}' is proved. Finally, the author applies his results to certain C_6' on Veronese's surface and thus obtains theorems on the sextactic points of curves of conical order 6 in the projective plane. *P. Scherk* (Saskatoon, Sask.).

Lauffer, R. Eine Vektorgleichung der Raumkurven n-ter Ordnung im euklidischen R_r. Abh. Math. Sem. Hansischen Univ. 15, 82–84 (1943). [MF 15829]

Given an algebraic curve $\mathfrak{x}=\mathfrak{x}(s)$ of order n in complex Euclidean r-space R_r (s denotes arc length; thus $\mathfrak{x}'^2=1$). If an R_{r-1} perpendicular to a vector $\mathfrak{a}$ intersects the curve in the n different points $\mathfrak{x}_i$, then

$$\sum_1^n((\mathfrak{a}\mathfrak{x}_i')\mathfrak{x}_i''-(\mathfrak{a}\mathfrak{x}_i'')\mathfrak{x}_i')/(\mathfrak{a}\mathfrak{x}_i')^3=0.$$

The case $r=2$ was proved by M. Reiss [cf. S. Lie, Gesammelte Abhandlungen, vol. 1, Oslo-Leipzig, 1934, pp. 461, 837]. The author derives the general formula from this special case by projecting the R_r into planes through $\mathfrak{a}$.

P. Scherk (Saskatoon, Sask.).

Hjelmslev, Johannes. Die Geometrie der schwachen Figuren. Danske Vid. Selsk. Math.-Fys. Medd. 20, no. 21, 64 pp. (1943). [MF 15395]

A weak figure in real projective 3-space is defined by a variable set of planes, lines, and points such that all the planes approach a single fixed plane, all the lines a single fixed line and all the points a single fixed point. The study of weak figures consists of the determination of the ratios of the infinitesimal volumes, surfaces, lengths and angles which they define. All the weak figures of this article are defined by means of tangents and secants of a plane or space arc in projective space whose points of contact and intersection all approach a specified end point of the arc.

The author first considers plane convex arcs OL with continuously turning tangents for which an index exists. If the tangent with a point of contact B intersects the tangent at O in Q then $\lim_{B\to 0} QB/OQ$ is defined to be the index of the arc at O. The index is independent of the choice of the coordinate system. A complete analysis is given of some weak figures defined with the help of two arcs which form a cusp of the second type at their common end point O. A result of this analysis is that the index of the outer arc of the cusp is at most equal to that of the inner arc.

Let AL be an arc in 3-space. Then λ is defined to be the index for A of a projection of AL from a point not in the osculating plane of A, μ to be the index for A of a projection of AL from A and ν to be the index for A of a section of the tangent surface of AL by a plane not containing A. For an arc AL for which λ, μ, ν exist, the indices for points corresponding to A of all possible plane arcs which are either projections of AL or plane sections of its tangent surface are computed in terms of λ, μ and ν by an analysis of a plane weak figure.

Again let OL be a convex plane arc with a continuously turning tangent. A special affine system of coordinates is chosen so that the arc lies in the positive quadrant, O being the origin and the x-axis the tangent at O. If for all point pairs (x, y), (x_1, y_1) a constant r, $r>0$, exists depending on the curve alone so that

$$\lim_{x\to 0,\, x_1\to 0} y_1/y = \lim_{x\to 0,\, x_1\to 0} (x_1/x)^{r+1},$$

the arc is said to have an exponent of curvature r. This number is independent of the choice of the coordinate system. Such an arc is proved to have an index $1/r$. Let O be the common end point of two arcs both with exponent of curvature r which form a cusp of the second type. Let OA_1A be a secant cutting the cusp arcs in A_1 and A. Then $\lim_{A\to O} OA_1/OA$, which is again independent of the choice of coordinates, is known as the ratio of curvature i_1. For cusps for which i_1 exists some similar ratios are defined and expressed in terms of i_1. A typical result is as follows. Let AA_1B be a straight line which cuts the cusp arcs in A and A_1 and the common tangent in B. If B does not approach O then $\lim_{A\to 0} OA_1/OA = i_1^{r/(r+1)}$. Some analogous results are given for the cases where the two arcs unite to form the other types of singularity or a regular arc.

Let AL be a 3-space arc. A special affine system of coordinates is chosen so that the arc is in the positive octant with A at the origin, the x-axis the tangent at A and the plane $z=0$ the osculating plane at A. Let (x, y, z) and (x_1, y_1, z_1) be the coordinates of any two points which converge to A. Then if for all such point pairs

$$\lim_{x\to 0,\, x_1\to 0} y_1/y = \lim_{x\to 0,\, x_1\to 0} (x_1/x)^{r+1},$$

$$\lim_{x\to 0,\, x_1\to 0} z_1/z = \lim_{x\to 0,\, x_1\to 0} (x_1/x)^{r+s+1}, \quad r>0,\ s>0,$$

the numbers r and s are defined to be the exponents of curvature and torsion, respectively, for the arc at A. A curve with exponents r and s is shown to have index numbers λ, μ, ν. Relations involving these five constants are given. The author studies curve singularities composed of two of the above arcs both in the positive octant and with common exponents r and s. The ratio of curvature I_1 for such a singularity is defined to be the ratio of curvature i_1 for the projection from any point not on the osculating plane at A. Let B_1, B be the intersections of the arcs with a plane containing the tangent at A. The limit $\lim_{B\to A} AB_1/AB$ is defined to be the ratio of torsion I_2. By the use of weak figures the author obtains expressions for the curvature ratios of all possible projections of the singularity and the plane sections of its tangent surface in terms of I_1 and I_2.

The paper closes with a suggested method of obtaining the analogous n-space results and with a remark that the basic idea of weak figures could be extended to nonrectilineal figures. *D. Derry* (Vancouver, B. C.).

Turri, Tullio. Sugli automorfismi del gruppo delle omografie. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 19–32 (1944). [MF 16226]

Improving and extending a result of É. Cartan [Leçons sur la Géométrie Projective Complexe, Gauthier-Villars, Paris, 1931], the author proves that in a complex projective space of any number of dimensions the automorphisms of the group of homographies are only those arising from projectivities and antiprojectivities. The extension is to any number of dimensions; the improvement is the suppression of the word "continuous" before the word "automorphisms." Likewise, the assumption of continuity in Cartan's statement of the fundamental theorem of complex projective geometry [op. cit.] can be dropped and the theorem stated in any number of dimensions. *J. L. Vanderslice.*

Hamada, T. Ein Satz in der projektiven Geometrie. Tôhoku Math. J. **49**, 112–113 (1942). [MF 14701]

Let P, Q, R be three triangles polar for a given conic. Consider the three conics passing through the vertices of the pairs of triangles (P, Q), (Q, R), (R, P). The author shows that the three points which these three conics taken in pairs have in common (besides the vertices of P, Q, R) are collinear. Two special cases are pointed out.

N. A. Court (Norman, Okla.).

Kadeřávek, Fr. Über die Fläche $z=\sin x\cdot\sin y$. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1939**, 3 pp. (1939). (Czech. German and French summaries) [MF 16124]

In der obigen Arbeit ist die Untersuchung der Schiebungsfläche $\sigma\equiv z=\sin x\cdot\sin y$ auf einem rein synthetischen Wege durchgeführt. Die beiden erzeugenden Kurven der Fläche sind die Sinuslinien; durch ihre Form nähert sich die Fläche den Biegungsflächen der Betondecken.

Author's summary.

Kadeřávek, Fr. Sur la généralisation des surfaces de révolution. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1939**, 3 pp. (1939). (Czech. French summary) [MF 16125]

Etant données deux courbes homologues M, N pour les axes ${}^1O\perp{}^2O$, on peut construire sur les cordes joignantes les points conjugés des courbes M, N dans les plans du faisceau ${}^2O$ les courbes $A, B, C, \cdots$ aussi homologues pour les axes ${}^2O\perp{}^1O$. Les courbes $A, B, C, \cdots$ font une surface—une généralisation des surfaces de révolution, ressemblante au bracelet, armilla en latin, et c'est pour cela qu'on peut appeller cette surface un armilloid. Cette surface est munie de deux systèmes des courbes homologues entre eux le long desquelles touchent la surface les cônes ayant les sommets sur les axes ${}^1O$, ${}^2O$. On peut choisir les courbes M, N dans les deux branches d'une même courbe ou remplacer une de ces courbes par une droite.

Author's summary.

Hamada, Takashi. Elementary modifications of Rogers' and Aiyar's theorems. Tôhoku Math. J. **49**, 114–118 (1942). [MF 14702]

Aiyar's theorem, that the orthopolar circle of a line x, which touches a conic confocal with another conic touching the three sides of a given triangle, with respect to the triangle cuts the joint-director circle of those two conics orthogonally, is derived by the author from the following proposition. A triangle $A_1A_2A_3$ and a line x are given in the plane. From any two points P, Q on x draw perpendiculars PM_1, PM_2, PM_3 and QN_1, QN_2, QN_3 on the sides of $A_1A_2A_3$, respectively. There exists a circle S which cuts the three circles, whose centers are N_1, N_2, N_3 and respective radii N_1M_1, N_2M_2, N_3M_3, orthogonally. Then S cuts the orthopolar circle of x with respect to $A_1A_2A_3$ orthogonally. The latter proposition is proved analytically.

N. A. Court.

Gambier, Bertrand. Configurations récurrentes. Ann. Sci. École Norm. Sup. (3) **61**, 199–230 (1944). [MF 14649]

Richmond's paper on Cox's chain theorem [J. London Math. Soc. **16**, 108–112 (1941); these Rev. **3**, 87] leads the author to the observation that many alternating chain theorems exhibit a common pattern. Such a theorem involves two kinds of elements, say points and planes or lines and congruences; they may, for convenience, be called even and odd elements. The union of two elements of different parity is then defined, for example, a point lying in a plane. The chain starts with a single even element O and the odd elements $a_1, \cdots, a_n$ which are united to O and play the same role with respect to O. Let a_ia_j be the unique element united to a_i and a_j or an element among those that are united to both a_i and a_j (in the case of Cox's chain, a point on the line common to the planes a_i, a_j). It is either verified or proved, as the case may be, that a unique odd element $a_ia_ja_k$ exists which is united to a_ia_j, a_ja_k, a_ka_i; analytically, this amounts to showing that three equations with three unknowns have a unique solution.

The critical point in forming the alternating chain is whether the four odd elements $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$, $a_1a_2a_3$ determine a unique even element united to all four of them. This is, in general, equivalent to discussing the solution of a system of equations whose number exceeds the number of unknowns involved. If the system is inconsistent, the chain breaks down; if the system is consistent and has a unique solution, we have an alternating chain theorem of Cox's type, which may be derived from Cox's by substituting for the terms point and plane of Cox the names of the even and odd elements used in the given theorem. A considerable number of such alternating chain theorems are considered, like the Miquel-Clifford chain, the axial congruences of Richmond, etc. The author also considers some nonalternating chains.

N. A. Court (Norman, Okla.).

Thébault, V. Sur la géométrie du triangle et du tétraèdre. Mathesis **54**, supplement, 49 pp. (1941). [MF 15550]

This is a collection of seven articles bearing the titles: (I) Points remarquables du triangle et du tétraèdre [pp. 1–9]; (II) Sur les cercles et les sphères d'Adams [pp. 9–21]; (III) Cercles et sphères du triangle et du tétraèdre [pp. 21–28]; (IV) Sur le cercle de Nagel [pp. 28–37]; (V) Sur les triangles et les tétraèdres antipodaires [pp. 37–40]; (VI) Sphères et points associés au tétraèdre [pp. 40–47]; (VII) Sur un tétraèdre spécial. (I) is an expansion of a paper reviewed before [C. R. Acad. Sci. Paris **212**, 327–328 (1941); these Rev. **3**, 86]. (II) A later paper on the same subject has already been quoted [Amer. Math. Monthly **49**, 170–173 (1942); these Rev. **3**, 251].

(III) The circles considered are each tangent to two sides of a triangle (T) and pass through the Nagel point P of (T) or the associates of P for (T). Analogous considerations are applied to a special tetrahedron (T), in which the lines, joining the vertices to the internal points of contact of the respectively opposite faces with the escribed spheres belonging to the truncs, are concurrent. The paper ends with a note by R. Bouvaist dealing with the special tetrahedron in which $\cos a+\cos a'=\cos b+\cos b'=\cos c+\cos c'$, where a, a'; b, b'; c, c' are the pairs of opposite dihedral angles of the tetrahedron.

(IV) Most of the known properties of the Nagel circle are proved anew and some new properties pointed out. (V) The lines joining a point D to the vertices of a triangle (T) meet the circumcircle of (T) in the vertices of a second triangle (T'). The antipedal triangles of D for (T) and (T') are considered and some properties of the analogous figure for the tetrahedron are given.

(VI) Spheres are considered having for diameters the segments joining points situated on pairs of opposite edges of a tetrahedron and dividing the edges in the same ratio. Also the volumes of the tetrahedrons formed by the quadritangent centers of a given tetrahedron are computed. "Note sur un faisceau tangentiel de quadriques associées à un tétraèdre quelconque" by R. Bouvaist completes this paper.

N. A. Court (Norman, Okla.).

Haarbleicher, André. Cubiques auto-inverses isogonales par rapport à un triangle. Ann. Fac. Sci. Univ. Toulouse (4) 4, 65–96 (1940). [MF 15167]

The cubics which are invariant under the isogonal transformation with respect to a triangle are divided by the author into three groups: (a) the cubics having a double point at a vertex of the basic triangle, passing through a second vertex and through the two tritangent centers (that is, the centers of the circles touching the sides of the triangle) located on a bisector issued from the latter vertex; (b) the cubics circumscribed about the basic triangle; (c) the cubics circumscribed about the basic triangle and passing through the four tritangent centers of the triangle.

Numerous properties of each group of cubics are given. Thus a cubic of the group (c) is the locus of the pairs of isogonal conjugate points situated on a variable line revolving about a fixed point, the pivot, which lies on the cubic. The tangents to the cubic at the tritangent centers pass through the pivot. The tangents at the vertices of the basic triangle and the tangent at the pivot meet in the isogonal conjugate of the pivot, etc. The author points out that to this group belong five known cubics (Thomson's, Darboux's, etc.) which he identifies by their pivots. To these may be added the Neuberg cubic whose pivot is the point at infinity of the Euler line of the basic triangle [Moore and Neeley, Amer. Math. Monthly 32, 241–246 (1925)].

N. A. Court (Norman, Okla.).

Goormaghtigh, R. Pairs of triangles inscribed in a circle. Amer. Math. Monthly 53, 200–204 (1946). [MF 16145]

Stašek, Pavel. Analytischer Beweis des Chaslesschen Satzes (sowie des dualen Satzes) für die kubische Raumkurve. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1942, 7 pp. (1942). (Czech. German summary) [MF 16116]

Rao, C. V. H. On the Petersen-Morley theorem. Bull. Calcutta Math. Soc. 37, 131–132 (1945). [MF 16161]

Kubota, Tadahiko. Ein Beweis von der Bricardschen Verallgemeinerung des Feuerbachschen Satzes. Tôhoku Math. J. 48, 75–77 (1941). [MF 16351]

Osman, Ibrahim Ahmed. The isometric representation of the four dimensional Euclidean space on the flat. Proc. Math. Phys. Soc. Egypt 2, no. 2, 31–39 (1944). [MF 16338]

The four dimensional Euclidean space R_4 is mapped upon a hyperplane R_3. Four mutually perpendicular coordinate axes are selected in R_4 and the image-hyperplane is assumed to form equal angles with the four axes. Points, lines and planes are represented by their orthogonal projections and by the orthogonal projections of their auxiliary views. The auxiliary views are orthogonal projections on one of the hyperplanes formed by three of the coordinate axes. Hyperplanes are represented by projections of four points, preferably their traces. Some problems of position as well as some metric problems are discussed. *E. Lukacs.*

Pratelli, Gino. Sull'errore di inclinazione del fotogramma nella triangolazione radiale con immagini dell'orizzonte. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 78, 3–21 (1943). [MF 16238]

In aerial surveying of chains of rhombi or triangles a systematic error is introduced through the inclination ν of the plane of the plate to the horizontal terrain surveyed. The author shows that the common practice of neglecting an error due to a constant inclination ν is justified as this error is at most of the order ν^2. *E. Lukacs.*

Algebraic Geometry

Bydžovský, B. Über eine ebene Konfiguration $(12_4, 16_3)$. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1939, 8 pp. (1939). [MF 16112]

A $(12_4, 16_3)$ configuration consists of twelve points lying by threes on sixteen lines. The four lines through any one of the points, say A, contain two further points each; so there remain three points not joined to A. In both the $(12_4, 16_3)$ configurations that were considered by de Vries [Acta Math. 12, 63–81 (1889), in particular, p. 67] these three points are never joined to one another. The present author describes a new configuration in which four particular points are mutually separated, as before, whereas if A is any one of the remaining eight points, the three points not joined to A are joined to one another, forming a triangle of the configuration. He computes elliptic parameters for twelve such points on a general plane cubic curve, and deduces a construction by means of tangential points, one of the twelve points being arbitrarily assigned on the curve. Finally, he obtains coordinates for the general configuration of this type and deduces that the twelve points necessarily lie on a cubic. *H. S. M. Coxeter* (Toronto, Ont.).

Metelka, Josef. On certain $(12_4, 16_3)$ configurations in the plane. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1944, 8 pp. (1944). (Czech) [MF 16111]

The parameters of Bydžovský's twelve points on a plane cubic [see the preceding review] are denoted by u_i, v_i, w_i $(i=1, 2, 3, 4)$, where the v's are the four mutually separated points. The author considers two further points, having specified parameters v_5, v_6, and shows that any four of the six v's may be taken with the u's and w's to form a $(12_4, 16_3)$ configuration. Of the fifteen configurations so obtained, one is of de Vries's first type, two are of de Vries's second type, four are of Bydžovský's type and the remaining eight are of a new type. *H. S. M. Coxeter* (Toronto, Ont.).

van der Woude, W. On Cayley's solution of Poncelet's problem of closure. I. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 226–235 (1944). (Dutch. German, English and French summaries) [MF 15786]

If an $n$gon is inscribed in a conic V and circumscribed about a conic U, then the two conics are so related that there are infinitely many such $n$gons. This famous porism of Poncelet [Traité des Propriétés Projectives des Figures, 2d ed., vol. 1, Paris, 1865, p. 349], elegantly proved by Hurwitz [Math. Ann. 15, 8–15 (1878)], suggests the problem of expressing the relation between U and V in terms of their mutual invariants. It was shown by Cayley [Philos. Trans. Roy. Soc. London 151, 225–239 (1861)] that the desired condition is the vanishing of a determinant whose elements are the coefficients of $k^2, k^3, \cdots, k^{n-1}$ in the expansion of the square root of the discriminant of $U+kV$. The author gives a purely algebraic proof of this remarkable formula. (Cayley used elliptic integrals.)

H. S. M. Coxeter (Toronto, Ont.).

van der Woude, W. On Cayley's solution of Poncelet's problem of closure. II. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 375–379 (1944). (Dutch. German, English and French summaries) [MF 15789]

Cayley's own proof of his determinantal formula [see the preceding review] is given in modern notation. When thus expressed in terms of the Weierstrass $\wp$-function, it is substantially shortened. Incidentally, whereas the author claims to have corrected Cayley's errors, he introduces at least one of his own: $B'y^2$ for $2B'x$ in the eighth line.

H. S. M. Coxeter (Toronto, Ont.).

Bottema, O. Schläfli's theorem on polar simplexes. Nederl. Akad. Wetensch., Proc. 48, 499–504 = Indagationes Math. 7, 87–92 (1945). (Dutch) [MF 15801]

If two simplexes, in projective n-space, are reciprocal with respect to a quadric, so that each vertex of either is the pole of a bounding hyperplane of the other, then the $n+1$ lines joining corresponding vertices are associated, in the sense that every $(n-2)$-space which meets n of them meets the remaining one also. This theorem has been proved in various ways by Schläfli [J. Reine Angew. Math. 65, 189–197 (1866)], Baker [Proc. Cambridge Philos. Soc. 32, 507–520 (1936)] and Todd and Coxeter [Amer. Math. Monthly 51, 599–600 (1944)]. The present paper deals with further properties of these simplexes, especially in the three-dimensional case (where the four lines are generators of a regulus). *H. S. M. Coxeter* (Toronto, Ont.).

Bottema, O. An involution of lines in space. Nederl. Akad. Wetensch., Proc. 48, 505–512 = Indagationes Math. 7, 93–100 (1945). (Dutch) [MF 15802]

A variable line l meets two generators of each of two given reguli R_1 and R_2. These four generators have a second transversal l'. The correspondence between l and l' was investigated by de Vries, Schaake and Bone. They found a quartic congruence of singular lines, each of which corresponds to a flat pencil. The present paper is an improved treatment, using Klein's representation of lines by points on a quadric in 5-space. The following is typical of the results obtained. Let S_1 and S_2 be the associated reguli of R_1 and R_2, respectively. If l generates a ruled surface of order n, such that a generators belong to S_1, b to S_2, and c to the congruence of singular lines, then l' generates another ruled surface whose corresponding properties n', a', b', c' are given by $n'=3n-2a-2b-c$, $a'=2n-a-2b-c$, $b'=2n-2a-b-c$, $c'=c$.

H. S. M. Coxeter (Toronto, Ont.).

Bottema, O. On the axial surface of a pencil of linear complexes. Nederl. Akad. Wetensch., Proc. 48, 513–516 = Indagationes Math. 7, 101–104 (1945). (Dutch) [MF 15803]

A line l is called an axis of a given linear complex if all lines of the complex that meet l are perpendicular to l. In Euclidean space, any nonspecial linear complex has one axis and the axes of a pencil of linear complexes (containing a common linear congruence) generate a ruled cubic surface, Cayley's cylindroid. In elliptic space, a linear complex has two (polar) axes and the axes of a pencil of linear complexes generate a ruled quartic surface. [Clebsch and Lindemann, Vorlesungen über Geometrie, vol. 2, Leipzig, 1891, pp. 343–356.] It was proved by Appell [Bull. Soc. Math. France 28, 261–265 (1900)] that the pedal curve of any point (in Euclidean space) with respect to the generators of Cayley's cylindroid is a plane curve (namely, an ellipse). The author uses Jacobian elliptic functions to establish the corresponding property of Clebsch's ruled quartic surface in elliptic space: the pedal curve of a point P is a plane curve if and only if P lies on the surface. *H. S. M. Coxeter.*

Bottema, O. On the projective differential geometry of the ruled surfaces in R_4. XIV. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 146–152 (1943). (Dutch. German, English and French summaries) [MF 14310]

[The original was incorrectly numbered XII by a misprint, corrected in the paper reviewed below. Part XIII, by Bos, appeared in Nederl. Akad. Wetensch., Proc. 45, 669–674 (1942); these Rev. 6, 105.] This work illustrates the theory of a ruled surface in four dimensions, developed by Weitzenböck and Bos in earlier numbers of the present series, by taking a special case. Two planes V_1 and V_2 in [4] meet in one point S. Tangents are drawn from S to touch a fixed conic k_1 of V_1 at A_1 and B_1, and tangents from S to touch a fixed conic k_2 of V_2 at A_2 and B_2. The point S does not lie on either conic, but the points of the conics are related by a given correspondence such that A_1, A_2 correspond but B_1 and B_2 do not. The line joining a pair of corresponding points then generates a ruled quartic surface F, so that the line A_1A_2 is a particular generator. The conics are merely two of ∞^1 conics which generate the surface, forming a network with the ∞^1 generating lines. Each line intersects each conic once, but the line A_1A_2, the torsal line alone, stands out specially. A pretty theory emerges owing to the reciprocity of F and its companion surface ϕ, a ruled quartic of the same type. The surfaces F and ϕ meet in the companion curve, which is a twisted cubic, and which relates in (1, 1) correspondence the generator and the conic which meet on this curve at this point.

H. W. Turnbull (St. Andrews).

Bottema, O. On the differential geometry of the ruled surfaces in R_4. XV. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 201–206 (1943). (Dutch. German, English and French summaries) [MF 14311]

[Cf. the preceding review.] This continues the theory of F, the ruled quartic surface in [4], when the generating line meets two fixed conics in related points as before, but neither pair A_1A_2, B_1B_2 is related. The surface has a network of ∞^1 lines meeting each of ∞^1 conics once. This more general surface has no torsal line but has one double point. The companion surface ϕ (the locus of the transversal line of three consecutive generators of F) is a sextic, and the companion curve (common to F and ϕ) is a rational normal quartic curve of [4]. The bi-line (the fifth associated line of four consecutive generators of F) generates a surface which specialises in this case to a plane pencil of lines through the double point of F, containing the two generators which necessarily meet at this double point. *H. W. Turnbull.*

Differential Geometry

★Blaschke, Wilhelm. Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. Band I. Elementare Differentialgeometrie. 3d ed. Dover Publications, New York, N. Y., 1945. xiv+322 pp. $3.50.

Photographic reproduction of the 1924 edition of volume 1 of the series Die Grundlehren der Mathematischen Wissenschaften, published by J. Springer, Berlin. A German-English index-glossary has been added.

Lüssy, Willi. Äquipotentialkurven und ihre Orthogonaltrajektorien. Elemente der Math. 1, 25–31 (1946). [MF 16376]

Der Zweck dieser Arbeit ist, Kurvenscharen und deren Orthogonaltrajektorien zu untersuchen, die sich in Bipolarkoordinaten durch die Gleichung $|f(u)\pm f(v)|=$konst. darstellen lassen. *Extract from the paper.*

Stašek, Pavel. Über eine im Polarkoordinatensystem gegebene ebene Kurve, welche aus zwei gegebenen Kurven durch die Beziehung $r=r_1r_2$ entsteht. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1944**, 8 pp. (1944). (Czech. German summary) [MF 16110]

Bompiani, E. Analisi dei flessi di specie superiore delle curve piane. Boll. Un. Mat. Ital. (2) 5, 156–168 (1943). [MF 16102]

A plane curve whose equation may be written in the form (*) $y=\sum_{i=0}^{n} a_i x^{k+i}$, $a_0\neq 0$, n finite or infinite, $k>2$ is said to have an "inflexion" at the origin. If $k=3$, the inflexion is ordinary; if $k=4$, the inflexion is said to be of the second kind. Using various types of osculating curves to represent the curve for the appropriate neighborhoods of the origin, canonical forms of (*) for $k=3$ and $k=4$ are found. The remainder of the paper is concerned with (*) with $k>4$, a general method for studying the curve being developed. A typical theorem may be stated as follows. The element E_{k+1} of the $(k+1)$th order of (*) determines on the tangent at O an invariant point O_1; the element E_{2k-2} of the $(2k-2)$th order determines a set of invariant points of which O_1 is the harmonic center with respect to O. For $k>4$ there are $k-4$ projective invariants. *V. G. Grove.*

Maxia, A. Configurazioni metricamente legate ad un punto cuspidale di una curva piana. Boll. Un. Mat. Ital. (2) 5, 189–196 (1943). [MF 16082]

The author studies the Euclidean invariant theory of a cuspidal element. The projective theory of a cusp has been studied by E. Bompiani [Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 3, 138–151 (1942), and later papers]. The conformal theory of irregular analytic elements has been developed by Kasner [Trans. Amer. Math. Soc. 16, 333–349 (1915)]. Under the group of arbitrary analytic transformations, a complete classification of irregular analytic elements, with respect to absolute invariants, was obtained by Kasner and DeCicco [Trans. Amer. Math. Soc. 51, 232–254 (1942); these Rev. 3, 306]. The author considers the cuspidal element $x^2=a_{03}y^3+a_{12}xy^2+\cdots$. By associating the involutes of circles and certain cubic curves, there are obtained geometric constructions for the first two Euclidean invariants a_{03} and a_{12}. These results are applied to twisted curves of space. *J. DeCicco* (Chicago, Ill.).

Chariar, V. R. Note on transversals which meet consecutive generators of a ruled surface at a constant angle. Bull. Calcutta Math. Soc. 37, 133–136 (1945). [MF 16162]

Sbrana, F. Sopra alcune proprietà delle superficie. Boll. Un. Mat. Ital. (2) 4, 147–157 (1942). [MF 16065]

The author considers some properties of a surface in ordinary space. Let Σ be a surface which possesses an elliptic point O. Construct the tangent plane α to Σ at O and let s be any straight line in α. Through s, pass a plane β on which the surface Σ cuts off an area σ. The author proves that, as β tends into coincidence with α, the straight line joining O with the centroid G of σ tends to the reciprocal of s with respect to the osculating paraboloid of Σ at O, possessing the common affine normal with Σ. Also the axes of inertia of the ellipse of inertia of the area σ, with respect to the centroid G, tend into coincidence with the principal directions of Σ at O. Finally, there is established a relationship between the moments of inertia of σ with respect to any line in β through the centroid G and the curvatures of the normal sections of Σ at O as β tends into coincidence with α. *J. DeCicco* (Chicago, Ill.).

Gambier, Bertrand. Sur les couples de surfaces applicables avec conservation des courbures principales. Systèmes cycliques. J. Math. Pures Appl. (9) 23, 249–304 (1944).

This paper is concerned with pairs of applicable surfaces S, S_1 having equal Gaussian and mean curvatures at corresponding points. The method of attack is classical. The minimal curves on S, S_1 are made parametric, the linear element on each being $ds^2=2F\,du\,dv$. The functions D, D', D'', $\delta, \delta', \delta''$, Δ, Δ' being defined by the formulas $D=|x_u, x_v, x_{uu}|$, $D'=|x_u, x_v, x_{uv}|$, $D''=|x_u, x_v, x_{vv}|$, $D=\delta F^2=\Delta F$, $D'=\delta' F^2$, $D''=\delta'' F^2=\Delta'' F$, the second fundamental form of S is $i(\Delta du^2+2\delta' F\,du\,dv+\Delta'' dv^2)$ and the curvatures of S are $K=\delta\delta''-\delta'^2$, $K_m=-2i\delta'$. Then S, S_1 are of the type and in the relation desired if the values of F, δ' and $\delta\delta''$ are the same for S, S_1 at corresponding points. The second fundamental coefficients of S, S_1 are related as follows: $\Delta_1=\Delta-1/U$, $\Delta_1''=\Delta''-1/V$, $\Delta U+\Delta'' V=1$, U and V being functions of u alone and v alone. By means of these fundamental relations it is shown how a continuous deformation in one parameter may be decomposed into three classes of deformations. These classes and the types of surfaces S, S_1 are studied in detail. *V. G. Grove* (East Lansing, Mich.).

Charrueau, André. Sur la déformation infiniment petite des surfaces. Bull. Sci. Math. (2) 69, 92–108 (1945). [MF 15417]

The purpose of this paper is to discuss a set of twelve surfaces considered by Darboux [Leçons sur la Théorie Générale des Surfaces, vol. 4, Gauthier-Villars, Paris, 1896, pp. 1–72] as they are related to the theory of the deformation of surfaces. These twelve surfaces may be paired so that the surfaces of each pair correspond with orthogonality of linear elements. Darboux also showed that these twelve surfaces may be grouped into three sets of four surfaces each such that the asymptotic nets on the surfaces of each set correspond, and correspond to a conjugate net with equal point invariants on each of the surfaces of a certain four of the remaining eight, and to a conjugate net with equal tangential invariants on each of the remaining four. Let $f_1, f_2, f_3, F_1, F_2, F_3$ be functions of two variables such that $df_i\,dF_i=0$. Then the four pairs of surfaces described by the points (f_1, f_2, f_3), (F_1, F_2, F_3); (F_1, f_2, f_3), (f_1, F_2, F_3); (f_1, F_2, f_3), (F_1, f_2, F_3); (f_1, f_2, F_3), (F_1, F_2, f_3) correspond with orthogonality of linear elements. This notion is applied to the study of the twelve surfaces of Darboux. Finally, the results are applied to pairs of surfaces which are representations of pairs of harmonic functions [A. Charrueau, Bull. Sci. Math. (2) 67, 168–176, 179–187 (1943); these Rev. 7, 77]. *V. G. Grove* (East Lansing, Mich.).

Fernandez Avila, Francisco Javier. Some properties of orthogonal nets. Revista Mat. Hisp.-Amer. (4) 5, 113–122, 164–182 (1945). (Spanish) [MF 15203]

If an ellipsoid with axes a_i of lengths A_i is given for every point of a domain in 3-space, as is the case in many mechanical problems, then the a_i will in general not be tangent to a triply orthogonal system of surfaces. Conditions were first given by Boussinesq and Weingarten. The present paper gives other conditions which generalize results of Lamé. Denote by c_i the curves with tangents a_i, with s_i the arc-length on c_i, and by ρ_1^3 the radius of curvature of the projection of c_1 on the plane a_1a_3. If T is the tensor corresponding to the quadratic form $\sum A_i a_i^2$, then the condition is

$$\operatorname{div} T = SA_1\left(\frac{\partial A_1}{\partial s_1}+\frac{A_1-A_2}{\rho_2^3}+\frac{A_3-A_1}{\rho_3^2}\right),$$

where S indicates summation over the cyclic permutations. For div $T=0$, known formulas of Lamé are obtained.

If $F(x, y)=c$ defines a family of curves in the plane $S=(F_x^2+F_y^2)^{\frac{1}{2}}$ and n is the normal of the curve of the family at a point, ρ^{-1} its curvature, then $\partial S/\partial n=\Delta F-S/\rho$. Various known results on orthogonal nets in a plane are derived from this formula. [Misprints and multivalued notations are numerous.] *H. Busemann.*

Lalan, Victor. Représentation conforme avec conservation des pseudo-arcs des lignes minima. C. R. Acad. Sci. Paris 222, 632–633 (1946). [MF 16043]

The problem at hand is the determination of those pairs of surfaces which can be put into conformal correspondence with the preservation of the pseudo-arcs of the minimal curves. This paper is an announcement of general results without detailed formulae or proofs. The general solution depends upon five arbitrary functions of a single argument. If the local ratio of similitude between the two surfaces is constant (not 1), the solution depends only on two arbitrary functions. The method of attack is based upon the fundamental quadratic form $ds^2=(2\omega_1\omega_2)/A$, in which ω_1 and $i\omega_2$ are, respectively, the differentials of the pseudo-arcs of the minimal lines. If A is a constant, the two surfaces are cylinders of revolution. If ω_1 and ω_2 are given, the surface is essentially determined except in certain ambiguous cases.

C. B. Allendoerfer (Haverford, Pa.).

Picasso, Ettore. Connessioni proiettive su una superficie di S_4. Rend. Sem. Fac. Sci. Univ. Cagliari 14, 14–18 (1944). [MF 16225]

A new and simple proof is offered of a theorem first enunciated by É. Cartan. In projective 4-space there exists a surface with arbitrarily assigned components of a normal projective connection without torsion. It is also shown how such a projectively connected surface determines on itself an intrinsic affine connection and an intrinsic affine covariant vector. *J. L. Vanderslice* (College Park, Md.).

Gheorghiu, Gheorghe Th. Sur une classe de surfaces. C. R. Acad. Sci. Paris 222, 357–359 (1946). [MF 16010]

This is a further discussion of a class of surfaces in projective space which arose in an earlier study of the author [Bull. Sci. Math. (2) 69, 12–20 (1945); these Rev. 7, 79]. Five geometric and analytic properties are stated without proofs. *J. L. Vanderslice* (College Park, Md.).

Hsiung, C. C. Projective invariants of contact of two curves in space of n dimensions. Quart. J. Math., Oxford Ser. 17, 39–45 (1946). [MF 15895]

Let C_1, C_2 be two curves in a projective space S_n which are tangent at a general point O, and suppose that the osculating plane of neither of them is contained in the osculating hyperplane of the other. Then the author constructs a set of invariants I_{ij} $(i\neq j;\ i, j=2, \cdots, n)$ which he characterizes geometrically in terms of certain cross ratios in a manner too involved to be described here.

J. E. Wilkins, Jr. (Chicago, Ill.).

Segre, B. On tac-invariants of two curves in a projective space. Quart. J. Math., Oxford Ser. 17, 35–38 (1946). [MF 15894]

The results of the paper reviewed above are extended to the case in which the curves C_1 and C_2 have their first r osculating linear spaces in common but have independent osculating linear spaces of dimension $r+1$. The author computes invariants I_{ij} $(i\neq j;\ i, j=r+1, \cdots, n)$ by means of a rather complicated system of cross ratios, and shows that these can all be expressed as functions of the $n-r-1$ independent invariants $I_{r+1,j}$ $(j=r+2, \cdots, n)$. For $r=1$, these invariants coincide with those of Hsiung, but the geometrical interpretation is somewhat simpler.

J. E. Wilkins, Jr. (Chicago, Ill.).

Chang, Su-Cheng. A new foundation of the projective differential theory of curves in five-dimensional space. Trans. Amer. Math. Soc. 59, 132–165 (1946). [MF 15320]

The first chapter of this paper is devoted to the development of a theory of plane curves. Upon the groundwork of this theory a theory of curves in five dimensional space is developed in the second chapter.

The canonical expansions of Halphen and Lane for the equations of a plane curve in the neighborhoods of (i) an ordinary point and (ii) a sextactic point, respectively, are obtained by effecting new geometric characterizations of the associated fundamental reference triangles. Canonical expansions for the equation of a plane curve at generalized sextactic points ($k$ic points, $k\geqq 6$) are then obtained. A point P on a plane curve C is called a $k$ic point if the osculating conic has k-point contact at P with the curve C. The associated reference triangles at the $k$ic points ($k\geqq 6$) are geometrically characterized as follows. (i) The 7ic point. There exists a pencil p_α of quartic curves (21′), each quartic Q_4 of which has contact of order eight with C at P and which has a cusp at $A(1, 0, \alpha)$ whose cuspidal tangent coincides with the tangent to C at $P(0, 0, 1)$. The polar line of A with respect to the osculating conic of C at P intersects Q_4 at three points, one point of which describes, as α varies, a curve (25) of degree six which intersects the osculating conic of C in four points at P and in two points on a line (26) which passes through P. The line (26) is chosen to be the edge $x=0$. The intersection of (26), distinct from P, with the osculating conic of C at P and the pole of (26) with respect to this conic are chosen as the vertices P_2 and P_1, respectively. The special form $y=x^2-2x^7/3+(9)$ of the canonical expansion finally results from the choice of the unit point on the osculating conic of C at P in such a manner that the point $(1, 1, 0)$ lies on the curve (25). (ii) The 8ic point. In this case there is a unique quartic curve of the pencil of quartics (21′) which has contact of order nine with C at P. The cusp of this quartic is selected as the vertex P_1, the polar line of P_1 with respect to the osculating conic of C at P is the edge $x=0$, and the point

P_2 is chosen to be the intersection, distinct from P, of the edge $x=0$ with the osculating conic of C at P. If the common points of $x=0$ and the quartic, having contact of order nine with C at P, are projected from P_1 to the osculating conic of C at P, six points are obtained. If any one of these points is chosen for the unit point, the canonical expansion assumes the form $y=x^3+x^6+(10)$.

In five dimensional space the osculating plane p at an ordinary point P of a curve Γ intersects the developable hypersurface of Γ in a plane curve C, of which P is either an ordinary point or a $k$ic point ($k=6, 7, 8$). For each of these cases a method described in the first chapter is available for the determination of a covariant unit point I and a covariant triangle $\{PP_1P_2\}$ in p, the line PP_1 being tangent to Γ at P. Let p denote a plane passing through PP_1 and lying in the osculating three space of Γ at P and let Γ_1 denote the plane curve in p obtained by projecting Γ from a point $P+\rho P_2$ (collinear with P and P_2). The Bompiani osculant O_8 associated with the point of inflexion P of Γ_1 describes a generator of a covariant quadric Q_3 as p rotates about PP_1; the generator varies over Q_3 as the point of projection moves along PP_2. The fourth vertex P_3 of the reference pyramid is the fourth vertex of the quadrilateral on Q_3 whose first three vertices are the points P_1, P_2, P_2. In a similar manner, on projecting Γ from a point $P+\rho P_3$ $(P+\rho P_4)$ to a plane p which passes through PP_1 and lies in the osculating four space (five space) of Γ at P, after imposing suitable conditions, the locus of the osculant O_8 (O_{10}) is found to be a covariant quadric Q_4 (Q_5). The vertex P_4 (P_5) is the fourth vertex of the quadrilateral $P\,P_1P_3P_4P$ $(P\,P_1P_4P_5P)$ which lies on Q_4 (Q_5).

The pyramid just described is valid only in case every point of Γ is an ordinary point of C. For each of the cases in which a point of Γ is a $k$ic point of C ($k=6, 7, 8, 9$), the determination of P_2 requires a particular modification. The author obtains in each specific case (1) the characterization of the associated reference pyramid, (2) corresponding canonical expansions for the equations of Γ and (3) related projective Frenet-Serret formulas. The canonical expansions and the Frenet-Serret formulas are developed in terms of coefficients of assumed power series in nonhomogeneous coordinates for the equations of Γ. *P. O. Bell.*

Hirakawa, Junkô. The relative differential geometry in affine space. Jap. J. Math. 17, 347–400 (1941). [MF 14963]

Let $x(t)=(x_1(t), x_2(t))$ traverse a plane curve with the unit normal $\xi(t)$, affine arc length $\bar{s}(t)$ and curvature $\bar{\rho}^{-1}(t)$. If $p(t)$ is the perpendicular distance from a fixed point a to the tangent of $x(t)$, then $P(t)=p\bar{\rho}^{\frac{1}{3}}$ is the affine perpendicular distance from a to $x(t)$, according to Süss's notation. If the convex curve E is the unit circle of a plane Minkowski geometry and $y(t)$ traverses E in such a way that tangents of E and x are parallel for corresponding values of t, then the relative affine (r.a.) arc length is defined by $s=gd\bar{s}$, where $g(t)$ is the affine perpendicular distance from a to $y(t)$. If $\xi_1=dx/ds$, $\xi_2=g^{\frac{1}{2}}d^2x/ds^2$, then the r.a. curvature is $K=(\xi_2, \xi_2')=g^2(\xi_2, \xi_1'')$. With these concepts much of the usual relative differential geometry carries over to the affine plane, for instance, Bouquet's formulae, various characterizations of r.a. circles, properties of r.a. evolutes and natural equations. If K is an analytic function $K(s)$ of s, then the curve is in general (that is, when E has no special properties) determined up to a translation.

With certain complications the theory carries over to space curves. The natural equations are again discussed; r.a. curvature and r.a. torsion determine the curve in general up to a translation. The r.a. torsion vanishes if and only if the r.a. principal normals are parallel to a fixed plane.

Let E be a closed convex surface in E^3 and $x(u)=(x_i(u_1, u_2))$, $i=1, 2, 3$, a surface in E^3 parametrized so that for equal u the tangent planes of E and x are parallel. If a subscript i denotes differentiation with respect to u_i and $g(u)$ the affine perpendicular distance of the origin from the tangent plane of E at u, put $L_{ik}=|x_1, x_2, x_{ik}|$,

$$X=(x_1\times x_2)\cdot g^{-1}(L_{11}L_{22}-L_{12}^2)^{-\frac{1}{4}}.$$

The (symmetric) tensor $g_{ik}=gXx_{ik}$ plays the role of the fundamental tensor. If, finally, D_{ik} denotes covariant differentiation with respect to this tensor and $\xi_{ik}=gD_{ik}x-g_ix_k-g_kx_i$, then the r.a. normal to x at u is defined as $y=\frac{1}{2}g^{\alpha\beta}\xi_{\alpha\beta}$ and satisfies the relation $y_i=g(X\cdot D_{ik}y)g^{kj}x_j$. With these definitions a theory of surfaces can be developed along the traditional lines. For instance, a line of curvature is a curve along which the r.a. normals to the surface form a developable surface and has properties similar to the classical case. Integrability conditions corresponding to the Gauss-Codazzi equations are obtained, but they are much too involved to be given here. Among special results are characterizations of r.a. spheres and properties of surfaces with constant r.a. breadth. *H. Busemann* (Northampton, Mass.).

Haantjes, J. Conformal differential geometry. V. Special surfaces. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 322–331 (1943). (Dutch. German, English and French summaries) [MF 15778]

This paper uses the methods developed by the author in four previous papers [Nederl. Akad. Wetensch., Proc. 44, 814–824 (1941); 45, 249–255, 836–841, 918–923 (1942); these Rev. 3, 189; 6, 21]. Some special surfaces in three-space are studied. Among the results are the following three theorems. (1) If a system of lines of curvature of a surface consists of Darboux curves, then it consists of circles. (2) A necessary and sufficient condition that a line of curvature is spherical is that the angle between the conformal osculating plane and the tangent plane is constant along the curve. (3) If one system of lines of curvature consists of conformal geodesics the other system consists of circles. There are also conditions that the normal circles of a surface all pass through a fixed point and studies of surfaces of which the normal circles form a normal congruence. *D. J. Struik* (Cambridge, Mass.).

Wrona, W. Eine Verallgemeinerung des Schurschen Satzes. Nederl. Akad. Wetensch., Proc. 44, 943–946 (1941). [MF 15761]

The scalar curvature of an "m-direction" at a point P of an n-dimensional Riemann space is defined as the scalar curvature at that point of a geodesic m-dimensional subspace tangent to the given "m-direction." [See J. Haantjes and W. Wrona, same Proc. 42, 626–636 (1939); these Rev. 1, 89.] In this paper the following generalization of Schur's theorem is proved. If the scalar curvature is the same for all m-directions at P, then it is independent of P. Under this condition it further follows that, when $m<n-1$, the Riemann space is of constant curvature; when $m=n-1$, the space is an Einstein space. *C. B. Allendoerfer.*

Ruse, H. S. A. G. D. Watson's principal directions for a Riemannian V_4. Proc. Edinburgh Math. Soc. (2) 7, 144–152 (1946). [MF 16210]

The paper by Watson referred to appeared in the same Proc. (2) 6, 12–16 (1939). The purpose of the present paper

is to analyze Watson's theory in relation with the work of others by using methods developed by the author in his study of the Riemann complex. These methods are summarized for the reader's convenience. It is found that Watson's principal directions are the nonnull Kretschmann-Struik directions [D. J. Struik, J. Math. Phys. Mass. Inst. Tech. 7, 193–197 (1928)] for the self-dual part of the Riemann tensor. In the Einstein case to which Watson actually confined himself the Riemann tensor is already self-dual. There are in all six sets of these directions.

J. L. Vanderslice (College Park, Md.).

Ruse, H. S. The five-dimensional geometry of the curvature tensor in a Riemannian V_4. Quart. J. Math., Oxford Ser. 17, 1–15 (1946). [MF 15891]

The author continues his investigation of the geometry of the Riemann curvature tensor in four dimensions initiated in his previous paper [Proc. Roy. Soc. Edinburgh. Sect. A. 62, 64–73 (1944); these Rev. 6, 106]. In each affine tangent space the components of this tensor determine a quadratic complex in the projective 3-space at infinity. In the associated Cayley 5-space there appear as fundamental configurations (1) the Cayley quadric, (2) a quadric determined by the metric tensor when viewed in the projective 3-space as the family of its tangents and (3) the quadric of the Riemann complex. The Segre characteristics of the quadric pencils $(3)+\lambda(1)$ and $(3)+\lambda(2)$ can be used as basis for a more detailed classification of the Riemann complex than heretofore. Considerable space is devoted to developing an effective formalism, strong use being made of Veblen's spinor notation. Finally, these methods are applied to find special results regarding the Riemann complex of spaces of class one, spaces of constant curvature, conformally flat spaces and the Schwarzschild space-time. *J. L. Vanderslice.*

Ruse, H. S. The Riemann tensor in a completely harmonic V_4. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 156–163 (1945). [MF 16206]

A Riemannian space V_n is completely harmonic if the Laplacian of the arc length s measured from any base point is a function of s alone. Copson and Ruse [same Proc. Sect. A. 60, 117–133 (1940); these Rev. 2, 20] found as necessary and sufficient conditions an infinite sequence of tensor relations on the metric and Riemann tensor, the first two of which (A, B) were algebraic. The main question at issue is whether every completely harmonic Riemann space is of constant curvature. The converse is true and the direct theorem has been proved for $n=2, 3$, for $n=4$ (signature ± 2) and for conformally flat spaces. In the present paper the author obtains all types of V_4 that are algebraically possible under conditions A, B and finds that when the signature is not ± 2 there at least is no algebraic necessity for the V_4 (and hence V_n, $n>4$) to be of constant curvature. The author considers of greater interest than his result the illustration it provides of the use of methods developed by him in recent papers on the Riemann complex [see, for example, the preceding review]. *J. L. Vanderslice.*

Lichnerowicz, André, et Walker, A. G. Sur les espaces riemanniens harmoniques de type hyperbolique normal. C. R. Acad. Sci. Paris 221, 394–396 (1945). [MF 14680]

A Riemannian V_n is called harmonic with respect to a point M_0 [Copson and Ruse, Proc. Roy. Soc. Edinburgh Sect. A. 60, 117–133 (1940); these Rev. 2, 20] if the associated Laplace equation admits a solution which is a function of s only, s being the geodesic distance measured from M_0, completely harmonic if it is harmonic with respect to every point, and simply harmonic [Walker, Proc. Edinburgh Math. Soc. (2) 7, 16–26 (1942); these Rev. 4, 171] if the associated Laplace equation admits the solution $s^{-(n-2)}$. Copson and Ruse conjectured that all completely harmonic spaces are of constant curvature. This is known to be true when $n=2$ or 3, and, for any n, when V_n is conformal to a flat space. In the present paper the authors prove that every completely harmonic space of normal hyperbolic type (that is, of signature $n-2$) is of constant curvature and that every simply harmonic space of this type is flat. The latter result also holds for spaces of elliptic type, as shown by Thomas and Titt [J. Math. Pures Appl. (9) 18, 217–248 (1939); these Rev. 1, 120]. [For other work on harmonic spaces, see Lichnerowicz, C. R. Acad. Sci. Paris 218, 436–437, 493–495 (1944); these Rev. 6, 216; and the papers of Ruse reviewed above.] *H. S. Ruse* (Southampton).

Nordon, Jean. Sur la solution élémentaire d'une équation aux dérivées partielles associée à un espace riemannien harmonique. C. R. Acad. Sci. Paris 219, 436–438 (1944). [MF 15276]

A Riemannian space V_n whose lineal element is $ds^2=g_{ij}dx^idx^j$ is called harmonic at a point P if Laplace's equation $\Delta_2 u=0$ admits a solution $u=f(s)$ which depends only upon the geodesic distance s from P to a current point of V_n. It is shown that a necessary and sufficient condition that the equation $\Delta_2 u=K(s)F(u)$ (where $K(s)$, $F(u)$ satisfy stated inequalities) admits a solution which is a function of s only is that the space V_n is harmonic at P.

A. Fialkow (New York, N. Y.).

Blaschke, Wilhelm. Ein Satz von Herglotz zur Geometrie Riemanns. Ann. Mat. Pura Appl. (4) 19, 251–256 (1940). [MF 15087]

Let P be a point of a Riemann space R_3. Assume that every pair of geodesics at P lies in some geodesic subspace S_2 of R_3. Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 73, 215–225 (1921)] has proved that at P the R_3 admits a 3-parameter group of rotations. In this paper the author reviews the properties of geodesic subspaces and gives a simple proof of Herglotz's theorem. An extension is made to Riemann spaces R_n with definite or indefinite metrics. In this case R_n admits at P a rotation group of $\frac{1}{2}n(n-1)$ parameters provided that any set of $n-1$ geodesics at P lies in some geodesic hypersurface S_{n-1} of R_n.

C. B. Allendoerfer (Haverford, Pa.).

Sugawara, Masao. On the theory of harmonic functions in the general Poincaré-space. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 5, 1–32 (1944). [MF 15696]

The space R in question is the space of all $m\times n$ matrices Z over the field of complex numbers such that $n(Z)\leqq 1$, where $n(Z)$ is the least upper bound of $\sum_j\sum_h\sum_k z_{jh}\bar{z}_{jk}x_h\bar{x}_k$ for all the n-vectors x satisfying $\sum x_i\bar{x}_i=1$. The displacements in R are given by the equations

$$w=(u_1z+u_2)(u_3z+u_4)^{-1},$$

where U_1, U_2, U_3, U_4 are matrices such that

$$\begin{pmatrix} U_1' & U_3' \\ U_2' & U_4' \end{pmatrix}\begin{pmatrix} E_m & 0 \\ 0 & -E_n \end{pmatrix}\begin{pmatrix} \bar{U}_1 & \bar{U}_2 \\ \bar{U}_3 & \bar{U}_4 \end{pmatrix}=\begin{pmatrix} E_m & 0 \\ 0 & -E_n \end{pmatrix};$$

E_n is the unit $n\times n$ matrix. A Riemannian metric is defined in R by the equation

$$ds^2=\mathrm{Sp}\, dZ(E_n-\bar{Z}'Z)^{-1}d\bar{Z}'(E_m-Z\bar{Z}')^{-1},$$

and this is invariant under displacements in R.

In this Riemannian space there exists a theory of harmonic functions. The author develops this theory using matrix notation. The Poisson integral, the Dirichlet integral and the Kronecker integral are obtained, and the main properties of harmonic functions in a Riemannian space, including the mean-value theorems, are established. [The reviewer has not been able to check all the details owing to his inability to obtain access to all the papers referred to, but the general argument is clear.] *W. V. D. Hodge.*

Levine, Jack. A replacement theorem for conformal tensor invariants. Tôhoku Math. J. **49**, 69–86 (1942). [MF 14697]

The author obtains a replacement theorem for conformal tensor invariants, showing that any conformal tensor invariant can be expressed in terms of the complete conformal curvature tensor and its covariant derivatives, use being made of a set of nontensor invariants which are the coefficients in the power series expansion of the conformal curvature tensor in conformal normal coordinates. The methods followed are analogous to those of T. Y. Thomas [Ann. of Math. (2) **28**, 549–561 (1927)] for the corresponding theorem for projective tensor invariants. *L. C. Hutchinson.*

Kosambi, Damodar. Sur la différentiation covariante. C. R. Acad. Sci. Paris **222**, 211–213 (1946). [MF 15991]

Taking first a generalized space of paths of the second order involving an absolute path parameter t, the author shows how to find for tensors of any weight a tensorial derivative with respect to t along a path and from it a covariant derivative. He asserts that a similar extension can be made to higher path spaces and spaces with k-spreads. Finally, given an infinitesimal transformation Xf in an affinely connected space, he exhibits an operator which applied to a tensor of any weight furnishes the corresponding infinitesimal transformation of the latter in tensorial form and reduces to X for the special case of a scalar.

J. L. Vanderslice (College Park, Md.).

Mikami, Misao. On parabolas in the generalized spaces. Jap. J. Math. **17**, 185–200 (1941). [MF 14956]

The generalized spaces in question are affinely connected, where the connection components involve the differentials of the coordinates. A generalized parabola is a curve in such a space which under the affine point displacements maps into a tangent space as a parabola. The differential equations of these curves have the same form as those of a family of paths of the third order. The question is answered as to when a family of parameterized paths of the third order can be regarded as parabolas of a generalized affine space; and similarly for the nonparameterized (projective) case. A certain contravariant vector, a concomitant of paths and connection, plays the deciding role. Two spaces with the same parabolas have autoparallels projectively related and from the class of affinely connected spaces with the same parabolas a certain invariant connection can be singled out. Simplifications arising when the original connection coefficients are point functions are discussed.

J. L. Vanderslice (College Park, Md.).

Hombu, Hitoshi, and Mikami, Misao. Parabolas and projective transformations in the generalized spaces of paths. Jap. J. Math. **17**, 307–335 (1941). [MF 14961]

In the paper by M. Mikami reviewed above, parabolas in an affinely connected space were studied as a specialized system of paths of the third order. The present paper first considers two such systems of parabolas in projectively related generalized affine spaces and proves the following theorem (well-known in classical affine geometry). Given a parabola of one system and a point thereon, there is just one parabola of the other system which is in third order contact at the point. It is shown also that through a given curve element of the second order there passes a parabola of each system in mutual fourth order contact. In the classical case such parabolas coincide. The remainder and major portion of the paper is devoted to the question: in what path spaces and under what projective transformations is it possible that any pair of parabolas of the two systems in fourth order contact at a point coincide? The problem is reduced to the solution of a system of differential equations. The solution is carried through completely for the case where one of the spaces is affinely flat and, for the general case, an enumeration of types is effected through a consideration of the canonical forms of a Pfaffian. A geometrically interesting special case of one of these types is developed in full. Finally, the differential equations are completely integrable only when the space is affinely flat or projectively flat of one specific type. *J. L. Vanderslice.*

Choquet, Gustave. Étude métrique des espaces de Finsler. Nouvelles méthodes pour les théorèmes d'existence en calcul des variations. C. R. Acad. Sci. Paris **219**, 476–478 (1944). [MF 15279]

Let $F_i(x, \xi)$, $i=1, 2$, be a continuous variational integrand with a convex indicatrix defined for $x \in D_i$, where D_i is an open domain in E^n. If there is a mapping ϕ of D_1 on D_2 which is isometric with respect to the Finsler metrics determined by the F_i, then the indicatrices at corresponding points of the D_i are linear transforms of each other. The mapping ϕ is continuously differentiable on the open subset G of D_1 where the indicatrix is not linearly equivalent to a surface of revolution; ϕ may be nondifferentiable in a dense subset of $D_1 - G$. If $F(x, \xi)$ is continuous in D, but the indicatrix not necessarily convex, $F^*(x, \xi)$ the integrand whose indicatrix is for every x the boundary of the convex closure of the indicatrix of F, then a shortest connection γ of two points a, b in D with respect to F^* has the following property with respect to F. In every neighborhood of γ there are arcs from a to b whose F-length is arbitrarily close to the greatest lower bound of the F-lengths of all arcs from a to b. [This result is contained in sections 3, 4 of H. Busemann and W. Mayer, Trans. Amer. Math. Soc. **49**, 173–198 (1941); these Rev. **2**, 225.] For the two-dimensional case generalizations of transversals and similar concepts are indicated. No proofs are given. *H. Busemann* (Northampton, Mass.).

RELATIVITY

Møller, C. On homogeneous gravitational fields in the general theory of relativity and the clock paradox. Danske Vid. Selsk. Math.-Fys. Medd. **20**, no. 19, 26 pp. (1943). [MF 15396]

The metric $ds^2 = dX^2 + dY^2 + dZ^2 - dT^2$ of space-time in the special theory of relativity can be transformed into $ds^2 = dx^2 + dy^2 + dz^2 - (1+gx)dt^2$, where g is any constant. The frame of reference with coordinates X, Y, Z, T is a system of inertia and that with coordinates x, y, z, t can be interpreted as having uniform acceleration g in the direction of

the X-axis; the transformation connecting the two systems is

$$x=-g^{-1}+g^{-1}\{(1+gX)^2-g^2T^2\}^{\frac{1}{2}},\quad y=Y,\ z=Z,$$

$$t=(2g)^{-1}\log\frac{1+gX+gT}{1+gX-gT}.$$

With this transformation it is possible to give a detailed explanation of the fact (at one time regarded as a paradox) that, if one of two identical clocks remains at rest in a system of inertia while the other, initially with the first, moves away from it and later returns to it, then the time which has elapsed according to the second clock is less than that measured by the first. *A. G. Walker* (Liverpool).

Depunt, J. Parametric representations of the Lorentz transformations. Wis- en Natuurk. Tijdschr. 12, 78–85 (1944). (Dutch) [MF 15588]

The author considers the linear transformations which leave $c^2t^2-x^2-y^2-z^2$ invariant (Lorentz transformations). Such a transformation depends on 6 parameters of which the geometrical meaning is given. Another form of the transformations is given by considering the two systems of E_2's on the light cone. *J. Haantjes* (Amsterdam).

Ives, Herbert E. Derivation of the Lorentz transformations. Philos. Mag. (7) 36, 392–403 (1945). [MF 15477]

The principles of conservation of momentum and energy are applied to the impact of a train of radiation on a perfectly reflecting particle. The usual relativistic variation of mass with velocity is deduced. Next, the same problem is studied with the system in uniform motion. The contraction of lengths and time intervals is derived and hence the Lorentz transformations. The author concludes: "The fact that the Lorentz transformations with all their consequences are deducible from [the laws of conservation] classes the 'Special Principle of Relativity' as a superfluous hypothesis." *A. Schild* (Toronto, Ont.).

Thomas, L. H. Relativistic invariance. Rev. Modern Phys. 17, 182–186 (1945). [MF 13688]

The author studies the general relativistic scheme of a dynamical theory in space-time. He considers how the specification of a "state" changes in passing from one member to another of a ten-parameter family of coordinate frames or "observers." "Displacement operators" are defined, whose forms are determined by the dynamical laws. The consistency of these laws requires that the physical quantities associated with a state return to their original values after a circuit of observers is traversed. The consistency conditions assume the form of commutability relations for the displacement operators. The author's general scheme can be specialized by the "relativistic" requirement that the ten-parameter family of observers reduce to a four-parameter set of six-parameter sub-families, each of the latter consisting of coordinate frames with a common origin. *A. Schild* (Toronto, Ont.).

Mosharrafa, A. M. The principle of indeterminacy and the structure of world-lines. Proc. Math. Phys. Soc. Egypt 2, no. 1, 27–29 (1941). [MF 14111]

It is assumed that the world-lines of material particles and photons have "discontinuities," characterized by $\delta(ds)=h/\mu c$. The de Broglie wave length and frequency and Heisenberg's principle of indeterminacy are obtained. *A. Schild* (Toronto, Ont.).

Vaidya, P. C. Spherically symmetric line-elements used in general relativity. J. Univ. Bombay (N.S.) 14, part 3, 4–6 (1945). [MF 15674]

The author studies coordinate transformations in t and r which transform the isotropic line element

$$ds^2=e^{\nu}dt^2-e^{\mu}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\},\qquad \nu=\nu(t,r),\ \mu=\mu(t,r), \tag{1}$$

into another line element of the same form. The conditions for the line element (1) to be stationary are that $1+\frac{1}{2}r\mu$ and $e^{-\frac{1}{2}\nu}\partial\mu/\partial t$ are functions of $re^{\frac{1}{2}\mu}$. *A. Schild.*

Einstein, A., and Straus, E. G. Corrections and additional remarks to our paper: The influence of the expansion of space on the gravitation fields surrounding the individual stars. Rev. Modern Phys. 18, 148–149 (1946). [MF 15757]

The paper appeared in the same Rev. 17, 120–124 (1945); these Rev. 7, 87.

Mautner, Friedrich, and Schrödinger, Erwin. Infinitesimal affine connections with twofold Einstein-Bargmann symmetry. Proc. Roy. Irish Acad. Sect. A. 50, 223–231 (1945). [MF 14283]

The main problem discussed is the determination of the general expression for an affinity which will carry both a nonsingular symmetric tensor field g_{ik} and a skew tensor field f_{ik} over into themselves. With this expression obtained, the authors make a suggestion for the generalization of the relativistic field equations. *M. Wyman.*

Infeld, L., and Schild, A. A new approach to kinematic cosmology. Phys. Rev. (2) 68, 250–272 (1945). [MF 14666]

Starting from the hypotheses that the velocity of light must have the same value everywhere, that space-time is spatially isotropic and homogeneous, the metric of space-time is shown to be conformal to a flat space-time. The theory leads to a certain restricted class of the expanding universes of general relativity. The properties of the geodesics and of the coordinate-transformations of these space-times are worked out in detail. *G. C. McVittie.*

Lichnerowicz, André. Sur le caractère euclidien d'espaces-temps extérieurs statiques partout réguliers. C. R. Acad. Sci. Paris 222, 432–434 (1946). [MF 16021]

The author proves the following theorem. In empty space a static space-time which is Euclidean at infinity cannot be regular everywhere without reducing to Euclidean space-time. *M. Wyman* (Edmonton, Alta.).

Lichnerowicz, André. Sur les équations relativistes de l'électromagnétisme. Ann. Sci. École Norm. Sup. (3) 60, 247–288 (1943). [MF 14643]

The first part of this paper deals mainly with the geometric significance of the various elements entering into the relativistic field equations of electromagnetism. This geometric significance is obtained by considering the principal directions at any point P of a Riemannian space whose metric is given by $ds^2=g_{ij}dx^i dx^j$. The determination of the principal directions depends on solving the determinant equation $|T_{ij}-sg_{ij}|=0$ for the four possible values of s, where T_{ij} is the energy-momentum tensor. The author shows that the values of s can be used to characterize the type of field (gravitational, electromagnetic or combination of both), with which we are dealing. General expressions T_{ij}, g_{ij} are obtained in terms of the four vectors which determine the principal directions.

In the second part of the paper the author extends some of the results of classical potential theory to relativistic theory. Existence theorems, uniqueness theorems, singularities, etc., are discussed for the gravitational field equations and the Maxwell-Lorentz equations. *M. Wyman.*

Dive, Pierre. Essai d'une théorie de la propagation ellipsoïdale des champs électromagnétiques et gravifiques. C. R. Acad. Sci. Paris 219, 235–237 (1944). [MF 15254]

The author discusses the relationships between the coefficients of the line element and the potential-tensor in order to obtain ellipsoidal propagation of the electromagnetic and gravitation fields. *M. Wyman* (Edmonton, Alta.).

Coleman, A. J. Phase space in Eddington's theory. Philos. Mag. (7) 36, 269–278 (1945). [MF 15749]

In his book, The Relativity Theory of Protons and Electrons [Cambridge University Press, 1936], Eddington introduces the concept of a 10-dimensional metrical "phase space" which is claimed to be closed and to have a finite volume. The author proves that the volume elements of "phase space" are singular on a 9-dimensional hypersurface and that consequently the volume of "phase space" is infinite. Incidentally, the author defines a "phase space" which is topologically and metrically equivalent to Eddington's but which, unlike the latter, is unique and not dependent on a generating "E-number." *A. Schild.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Arley, Niels. A note on the foundations of geometrical optics. Danske Vid. Selsk. Math.-Fys. Medd. 22, no. 8, 21 pp. (1945). [MF 15406]

The author's summary is as follows. "As is well known, all the laws of geometrical optics may be deduced from Fermat's principle or the equivalent principle of Huygens. It is the purpose of the present note to deduce these principles from Maxwell's electro-magnetic theory of light in the general case of non-ferromagnetic, absorbing, inhomogeneous, anisotropic media, at the same time obtaining the exact conditions for the validity of geometrical optics. In § 1 the problem is stated. In § 2 the equation for the phase function is deduced together with the conditions mentioned. In § 3 the phase equation is written in Hamiltonian form and Fermat's and Huygens' principles are deduced together with the generalizations of the normal and ray equations of Fresnel. It is seen that the n occurring in Fermat's principle is the ray index and not the index of refraction, as seldom stressed in the literature. Finally, the case of discontinuity surfaces is discussed."

The paper is a well presented development of the subject. It is regrettable that the author is obviously unaware [see § 2, p. 4] of the previous work of Debye, Picht, and Luneberg, which contains many of his results [Debye, Ann. Physik (4) 30, 755–776 (1909); Picht, Optische Abbildung, Vieweg, Braunschweig, 1931; Luneberg, Mathematical Theory of Optics, Brown University, Providence, R. I., 1944; these Rev. 6, 107]. *M. Herzberger.*

Maruyama, Shuzi. A theorem in geometrical optics. Proc. Phys.-Math. Soc. Japan (3) 23, 1010–1015 (1941). [MF 15015]

An optical system corrected for Seidel aberrations can be considered as a system composed of two components. The author investigates the conditions imposed on the partial system. His results can be described as follows. Let S_1, S_2, S_3', S_3'', S_4 be spherical aberration, coma, two astigmatic errors and distortion. Seidel has shown that, if $S_1=S_2=S_\nu=0$, we can find a stop position for which $S_{\nu+1}$ is corrected. The author proves that, if $S_1=S_2=S_\nu=0$ for the whole system, and the stop for correction of $S_{\nu+1}$ is adjusted so as to be conjugated for the two partial systems, then $S_{\nu+1}$ must be corrected for the combined system. *M. Herzberger.*

Glaser, W., und Lammel, E. Strenge Berechnung der elektronenoptischen Aberrationskurven eines typischen Magnetfeldes. Arch. Elektrotechnik 37, 347–356 (1943). [MF 15626]

The authors have extended the calculations of previous papers [Glaser, Z. Physik 117, 285–315 (1941); Glaser and Lammel, Ann. Physik (5) 40, 367–384 (1941); these Rev. 4, 32; 6, 109] for a magnetic lens whose field along the axis is expressed by $H(z)=H_0/(1+z^2)^2$. The authors have calculated the five aberration coefficients B, C, D, E, F appearing in geometrical optics and the three anisotropic errors c, e, f arising from the magnetic field alone. The eight errors, expressed as functions of both the field and the position of the aperture plane $z=z_a$, are given in explicit form. It is found that the spherical aberration coefficient B cannot be made to vanish for any position of the aperture plane, whereas the system is free of coma for only one position of this plane. The isotropic astigmatism is found to be a quadratic function of z_a, whose roots are real, and thus two positions of the aperture plane are free of astigmatism. The isotropic distortion coefficient E is a cubic function of z_a, at least one of whose roots is real, making the distortion vanish for at least one position of the aperture plane. The anisotropic coma and the anisotropic astigmatism cannot be made to vanish for any value of z_a. Finally, the meridional and sagittal image curvature and the astigmatic difference are found to be functions of the square of the field. The distance between the meridional and sagittal line of confusion is found to depend on both the isotropic and anisotropic astigmatism. *N. Chako* (Chicago, Ill.).

Hutter, R. G. E. The class of electron lenses which satisfy Newton's image relation. J. Appl. Phys. 16, 670–678 (1945). [MF 14089]

The author has obtained the solution of the following problem of electron optics: what types of electric and magnetic fields satisfy both Newton's image relation and the magnification formula? Starting with the paraxial differential equation of an electron optical system, $y''+\lambda(z)y=0$, where $\lambda(z)$ is a function of the electric potential $\varphi(z)$ and the magnetic field $H(z)$,

$$\lambda(z)=\tfrac{3}{16}\varphi'(z)/\varphi(z)+\{e/(8m)\}H^2(z)/\varphi(z),$$

the author derives rigorously the functional relation which $\lambda(z)$ must satisfy in order that both Newton's image relation and the magnification formula are fulfilled simultaneously. A solution of this functional relation is

$$N(t)=(1-Et+t^2)^{-q},$$

where t depends on the distance between any point on the z-axis and the first focal point $z=z_{f_0}$ and E on the distance between the two focal points of the electron lens. It is proved that, for values of the parameter $E\geqq 0$, other solutions are constant multiples of $N(t)$. If E is restricted to $0<E<2$, any other solution can be represented by

$$M(t)=N(t)\sum_0^\infty c_m\left\{\left(\frac{t-\alpha}{t-\beta}\right)^{mq}+\left(\frac{t-\beta}{t-\alpha}\right)^{mq}\right\},$$

where $\alpha=\frac{1}{2}E+(\frac{1}{4}E^2-1)^{\frac{1}{2}}$, $\beta=\frac{1}{2}E-(\frac{1}{4}E^2-1)^{\frac{1}{2}}$. If α, β, E and t are replaced by their values, the author derives a complicated expression for $\lambda(z)$. To satisfy the magnification formula, he obtains a relation for the ratio of the focal distances of the lens. This ratio must be independent of z. This limitation makes it impossible to have combined electron lenses satisfying both Newton's relation and the magnification formula; hence, a combined lens will not fulfill the relations simultaneously. The restriction $0<E<2$ means that the principal planes of all electron lenses expressed by $\lambda(z)$ are crossed.

The author gives a few types of electric and magnetic lenses which fulfill the requirements. One of these lenses, namely, the magnetic lens, $H(z)=H_0/(1+z^2)^2$, has been treated exhaustively by Glaser and by Glaser and Lammel [cf. the preceding review]. Finally, the author shows that his results are the same as those of Glaser and Lammel, although they used a different method. *N. Chako.*

Hutter, R. G. E. Rigorous treatment of the electrostatic immersion lens whose axial potential distribution is given by: $\varphi(z)=\varphi_0e^{K \arctan z}$. J. Appl. Phys. 16, 678–699 (1945). [MF 14092]

The lens described in the title belongs to the type which satisfies both Newton's image relation and the magnification formula. The paraxial differential equation for such an electron lens can be expressed in terms of trigonometric functions. The corresponding problem for a magnetic lens has been treated by Glaser and by Glaser and Lammel [cf. the second preceding review]. From the general solution of the paraxial equation the author obtains simple expressions for the focal lengths, which have minimum values for values of the parameter k which are roots of the transcendental equations $\tan \pi/(1+k^2)^{\frac{1}{2}}=\pm 3^{\frac{1}{2}}k$. These values are $f_0/a=.199$ and $f_i/a=-4.45$, respectively; a is a constant expressing the half width of the potential curve when the potential reaches half its maximum value. A comparison with a thin lens is given. Newton's relation, the magnification formula, the position of the principal planes and the chromatic and spherical aberrations are given. Finally, the author generalizes the discussion to the case where the field is asymmetric. *N. Chako* (Chicago, Ill.).

Goddard, L. S. Optical characteristics of a two-cylinder electrostatic lens. Proc. Cambridge Philos. Soc. 42, 106–126 (1946). [MF 15666]

Explicit formulae are derived for the focal length and the principal planes of an electrostatic lens. The lens is of a type often used in electron optical instruments. It consists of two equal semi-infinite cylinders placed so that their axes coincide and so that the ends are separated by a small gap. For the axial potential of such a lens an asymptotic formula is used [Bertram, Proc. I. R. E. 28, 418–420 (1940)]. The author shows that the associated differential equation of the paraxial electron paths can easily be solved by Picht's method of iteration. The convergence of the resulting sequence is so rapid that two terms are sufficient for practical purposes. The method of iteration may also be applied if the axial potential is given only numerically. The method is found to be powerful enough to give information about electron optical devices without any experimental work. The design of a lens thus can be carried out to a great extent in the office rather than in the laboratory. *R. K. Luneberg* (Hanover, N. H.).

Goddard, L. S. A note on the Petzval field curvature in electron-optical systems. Proc. Cambridge Philos. Soc. 42, 127–131 (1946). [MF 15667]

The Petzval curvature of an electron lens has been expressed by Glaser by a simple integral formula [Z. Physik 117, 285–315 (1941)]. This formula, which contains the electric potential and magnetic field strength on the axis, is used by the author to evaluate the Petzval curvature of special electron lenses which are of practical importance: purely magnetic lenses and electrostatic lenses of the two-cylinder type discussed in the paper reviewed above. *R. K. Luneberg* (Hanover, N. H.).

Sugiura, Yoshikatsu, and Suzuki, Shigeo. On the magnetic electron lens of minimum spherical aberration. Proc. Imp. Acad. Tokyo 19, 293–302 (1943). [MF 14826]

Sugiura, Yoshikatsu, and Suzuki, Shigeo. Note on the magnetic electron lens of minimum spherical aberration. Proc. Imp. Acad. Tokyo 19, 544–545 (1943). [MF 14854]

The improvement of electron microscopes of high magnification depends largely upon the reduction of the spherical aberration of such an instrument. In view of the fact that this aberration cannot be eliminated completely, the problem is to find electromagnetic fields in which the spherical aberration is reduced to a minimum. By confining the question to the third order aberrations a problem of variation is obtained for the integral expression which represents the third order spherical aberration [Scherzer, Z. Physik 101, 23–26, 593–603 (1936); Glaser, Z. Physik 116, 19–33 (1940)]. For the case of a purely magnetic lens, the authors solve the Euler equations of this problem by a method of approximation. Their conclusion is that an instrument designed on the basis of their solution will have a limit of resolution about one tenth of the value for the best electron microscope at present (about 2–6 Ångstroms). *R. K. Luneberg* (Hanover, N. H.).

Riabouchinsky, Dimitri. Sur l'explication mécanique des équations de Maxwell. C. R. Acad. Sci. Paris 221, 391–394 (1945). [MF 14679]

Riabouchinsky, Dimitri. Dynamique de l'éther. C. R. Acad. Sci. Paris 221, 432–434 (1945). [MF 14684]

The author describes a method of transforming the Maxwell field equations into those of hydrodynamics, and conversely. He introduces a scalar E equal to the magnitude of the electric vector and a vector $\mathbf{H}'$ whose magnitude is that of the magnetic vector, but whose direction for a plane wave is normal to those of the electric and magnetic vectors. These quantities satisfy the equations: $\operatorname{div} \mathbf{H}'=-(1/c)\partial E/\partial t$ and $\operatorname{grad} E=-(1/c)\partial \mathbf{H}'/\partial t$. He is also led to consider, as an alternative to Newton's second law, the relation $\mathbf{F}=\mathbf{m}\times\mathbf{J}$, where $\mathbf{F}$ and $\mathbf{J}$ are force and acceleration vectors and $\mathbf{m}$ is a vector whose magnitude is the mass and whose direction is normal to that of the velocity vector. *O. Frink.*

Plenario, Antonio. Sopra una estensione di un teorema del Larmor. Boll. Un. Mat. Ital. (2) 4, 172–174 (1942). [MF 16068]

Si estende un noto teorema del Larmor supponendo il campo magnetico agente sopra la particella elettrizzata variabile in intensità col tempo. *Author's summary.*

Liénard, A. Électrodynamiques de Lorentz et de Hertz et principe de la moindre action. Ann. Fac. Sci. Univ. Toulouse (4) 7, 71–98 (1945). [MF 16177]

The electrodynamics of Hertz and Lorentz are developed by means of the principle of least action, in contrast to the thermodynamical method used in author's earlier paper [same Ann. (4) 5, 1–48 (1941); these Rev. 7, 270]. In the usual formulation of the pure radiation field, the action function is taken to be $(B^2-E^2)/2$. In this paper additional terms are introduced to take account of magnetic induction, dielectric polarization and Ohmic current. *C. Kikuchi.*

Chang, T. S. Quantum electrodynamics with $\partial A_\mu/\partial X_\mu=0$. Proc. Roy. Soc. London. Ser. A. 185, 192–206 (1946). [MF 15598]

The present paper investigates further the method of quantizing the electromagnetic field discussed in the author's earlier note [same Proc. Ser. A. 183, 316–328 (1944); these Rev. 6, 224]. The proof for the relativistic invariance of the commutation relations is given. Variants of the Dirac-Fock-Podolsky equations [Phys. Z. Sowjetunion 2, 468–479 (1932)], which give the interaction of the individual electrons with the field, are derived. It is also shown how the present formulation can be applied to Dirac's new electrodynamics [same Proc. Ser. A. 180, 1–40 (1942); these Rev. 5, 277] to eliminate the longitudinal component of the field. *C. Kikuchi* (East Lansing, Mich.).

Rozovskiĭ, M. On the equations of the electromagnetic field in a conducting medium with magnetic aftereffect. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 14, 402–406 (1944). (Russian) [MF 15378]

A plane parallel isotropic sheet of thickness l extending in direction z with conductivity γ, dielectric constant ϵ, permeability μ and magnetic after-effect is considered. The latter is represented by a Volterra type expression finally simplified to the form

$$B(z,t)=\mu H(z,t)+\int_{t_0}^{t}\varphi(t-\tau)H(z,\tau)d\tau,$$

which may be related to experimental results. This leads to a Volterra integro-differential equation for the field H of the form

$$\partial^2u/\partial t^2+ku(z,t)+\int_{t_0}^{t}\psi(t-\tau)u(z,\tau)d\tau=a^{-1}\partial^2u/\partial z^2,$$

where $u=\exp(\frac{1}{2}bt/a)\cdot H$ and the constants a, b, k and the function ψ are related to γ, ϵ, μ and the after-effect function and its derivatives, respectively. The equation is solved for the boundary conditions $H(0)=0$, $H(l)=0$ and initial conditions $H(z,t_0)=F_1(z)$, $\partial H(z,t_0)/\partial t=F_2(z)$ by Fourier analysis and iteration, which in this case is uniformly convergent. A closed expression for $H(z,t)$ involving an expansion is thereby obtained. *H. G. Baerwald* (Cleveland, Ohio).

Aurell, Carl-Georg. Contribution to the theory of telephone cables with twisted conductor groups. Ericsson Technics no. 45, 42 pp. (1944). [MF 15745]

The author first outlines the steady state solution of the fundamental equations for a system of n parallel homogeneous conductors. Then he considers a twisted spiral conductor, whose line constants are variable, and shows that if the pitch of the spiral is short compared with the wavelength the spiral conductor may be replaced by a straight one having the same total inductance and capacitance uniformly distributed along the cycle. When several spiral conductors are involved the mutual couplings may be distributed similarly.

To simplify the treatment of a cable consisting of a large number of twisted conductors the idea of a "cyclical symmetry group" is introduced, and explicit formulas for the propagation constants and characteristic impedances of such a group are obtained. A short discussion of crosstalk regarded as due to irregularities in the cable construction is followed by a detailed analysis of the constants of a symmetrically fed twisted pair in free space. The mean mutual inductance between spiral conductors overaplane earth is also obtained and finally the author discusses the computation of the direct capacities of a twisted conductor group. *M. C. Gray* (New York, N. Y.).

Duschek, A. Stromkräfte zwischen parallelen Leitern von rechteckigem Querschnitt. Arch. Elektrotechnik 37, 293–301 (1943). [MF 15629]

The known formula for the force per unit length between two parallel linear conductors, carrying uniform currents I_1 and I_2 and at distance a apart, is generalized first to the case of two parallel plane strips and then to the case of parallel conductors of rectangular cross-section. In each case the new formula is obtained by double integration so that the final formulas are complicated. Suitable approximations are suggested, particularly for two equal rectangular conductors at equal heights. *M. C. Gray.*

Schwarzer, H. Einige Betrachtungen über den Zusammenhang der Stromverteilung und des Verlaufes der magnetischen Feldlinien. Arch. Elektrotechnik 37, 287–292 (1943). [MF 15628]

Es wird bewiesen, dass in einem geraden zylindrischen Leiter die magnetischen Feldlinien nur dann konzentrische Kreise sein können, wenn die Stromverteilung symmetrisch zur Drahtachse und der Leiterquerschnitt ein Kreis ist. Es wird gezeigt, dass wenn die magnetischen Feldlinien in einem Laplaceschen Feld nicht konzentrische Kreise, also die Orthogonalflächen keine Ebenen sind, der Feldvektorbetrag längs einer Feldlinie nicht mehr konstant ist. Der Zusammenhang in einem solchen Feld zwischen der Krümmung der Feldlinien und der mittleren Krümmung der Orthogonalflächen wird angegeben. Ferner werden diese Überlegungen noch auf ein quellenfreies Wirbelfeld ausgedehnt. *Author's summary.*

Robin, Louis. Sur un problème de propagation et de diffraction d'ondes électromagnétiques, à la surface de séparation de deux milieux. C. R. Acad. Sci. Paris 218, 989–990 (1944). [MF 15248]

In a previous note [same C. R. 218, 135–136 (1944); these Rev. 7, 177] the author discussed the diffraction of light, considered as a scalar phenomenon, at the interface between two media. He now considers the same problem for electromagnetic waves. Space is supposed to be filled by two homogeneous media; the dielectric constant, permeability and conductivity are ϵ_r, μ_r, $\sigma_r/4\pi$, where $r=1$ or 2 according as x is positive or negative. The values of the electric and magnetic vectors **E** and **H** are given everywhere initially; the problem is to find them at any subsequent time.

The solution of mixed hyperbolic problems of the types of Dirichlet and Neumann gives **E** and **H** in terms of two auxiliary functions and the continuity conditions at the interface provide two integrodifferential equations for these auxiliary functions. Details of the derivation and solution

of these equations are not given. The author concludes that his problem has a unique solution when $(\epsilon_1-\epsilon_2)(\mu_1-\mu_2)<0$ and when $\mu_1-\mu_2=0$ but no solution if $(\epsilon_1-\epsilon_2)(\mu_1-\mu_2)>0$.
E. T. Copson (Dundee).

Elsasser, Walter M. Induction effects in terrestrial magnetism. I. Theory. Phys. Rev. (2) 69, 106–116 (1946). [MF 15535]

This paper investigates the electromagnetic effects produced by motions in the earth's core, considered as a sphere of fluid metal. For this, the solutions of the vector wave equation $\nabla^2\mathbf{A}_0+k^2\mathbf{A}_0=0$ are considered. The three independent solutions, designated by the terms scaloidal ($\mathbf{U}=\mathbf{R}\nabla\psi$), toroidal ($\mathbf{T}=\nabla\psi\times r$), and poloidal ($\mathbf{S}=\mathbf{R}\nabla\times\nabla\times\mathbf{r}\psi$), where ψ is a scalar point function [see, for example, J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, chapter 7], are examined and the possibilities of using these solutions to describe the variations of the earth's magnetic field are discussed. *C. Kikuchi* (East Lansing, Mich.).

Cotte, Maurice. Propagation d'une perturbation dans un guide électrique. C. R. Acad. Sci. Paris 221, 538–540 (1945). [MF 15143]

The author gives approximate formulas for the propagation down a wave guide of a nonperiodic disturbance, analogous to a switching transient, of one of the six components of a wave in the guide. It is assumed that the disturbance is decomposable into waves all of the same type but of continuously varying frequency. These frequencies are below the cut-off frequency of the guide, so that the disturbance is attenuated. The formulas are stated without proof. *O. Frink* (State College, Pa.).

Malov, N. The electromagnetic waves in the conical waves guide. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 15, 389–391 (1945). (Russian. English summary) [MF 15382]

The author treats H and E waves in conical guides of rectangular cross-section in standard fashion by separation in spherical coordinates. The relations between the radial order n and the smallest stationary azimuthal opening φ_0 are given in a table. Another table gives the field concentration Q for standing waves between two consecutive radial nodes due to conicity; this is compared with the conditions met in circular cylindrical guides. *H. G. Baerwald.*

Kahan, Théo. Sur les valeurs propres multiples dans un guide d'onde électromagnétique. C. R. Acad. Sci. Paris 222, 380–381 (1946). [MF 16018]

Kahan, Théo. Réflexion d'une onde électromagnétique sur un disque logé dans un guide d'onde. C. R. Acad. Sci. Paris 222, 998–1000 (1946). [MF 16393]

Kahan, Théodore. Méthode de perturbation appliquée à l'étude des cavités électromagnétiques. C. R. Acad. Sci. Paris 221, 536–538 (1945). [MF 15142]

Assuming the existence of only TM modes of field distribution in a symmetrical cylindrical cavity, the author computes the influence of a slight radial distortion by means of the theorem of Green, and obtains for the variation of frequency the ratio of a line and surface integral.
E. Weber (Brooklyn, N. Y.).

Kahan, Théo. Calcul de la fréquence propre perturbée d'une cavité électromagnétique (déformation de frontière). C. R. Acad. Sci. Paris 221, 694–696 (1945). [MF 15160]

With the same assumptions as in the paper reviewed above, the author attempts a slight radial distortion $\Delta R(\varphi)$. Expanding the perturbed Bessel functions into Taylor series, he shows that the frequency variation is related to the mean value of the radius. He demonstrates the identity of the result with that obtained in his first paper. *E. Weber.*

Durand, Émile. Calcul du champ créé par le mouvement d'une charge électrique. C. R. Acad. Sci. Paris 219, 510–513 (1944). [MF 15284]

The author shows that the solutions of Maxwell's equations obtained by R. Reulos [Ann. Physique (11) 7, 700–789 (1937)] are equivalent to the series expansion of the retarded potentials of the electromagnetic field [cf. the three following reviews]. *C. Kikuchi* (East Lansing, Mich.).

Durand, Émile. Calcul du champ créé par le mouvement d'une charge électrique. C. R. Acad. Sci. Paris 219, 584–586 (1944). [MF 15294]

The electromagnetic field due to a charge in motion is calculated by a method developed in the note reviewed above. *C. Kikuchi* (East Lansing, Mich.).

Durand, Émile. Sur l'identité des séries de potentiels et des formules de Liénard-Wiechert. C. R. Acad. Sci. Paris 221, 349–351 (1945). [MF 15233]

It is shown that the series solution of Maxwell's equation discussed in an earlier note [see the second preceding review] can also be identified with the series expansion of Liénard-Wiechert potentials. *C. Kikuchi.*

Durand, Émile. Passage de l'intégrale des potentiels retardés aux formules de Liénard-Wiechert. C. R. Acad. Sci. Paris 222, 284–286 (1946). [MF 16003]

It is shown that the Liénard-Wiechert potentials [see, for example, W. Heitler, The Quantum Theory of Radiation, Oxford University Press, 1936, pp. 19–21] can be obtained from Lorentz's retarded potentials by transforming the variables of integration. *C. Kikuchi.*

Durand, Émile. Les effets optiques du mouvement rectiligne et uniforme d'une source en théorie électromagnétique de la lumière. C. R. Acad. Sci. Paris 221, 401–403 (1945). [MF 15231]

The method developed in the notes reviewed above is used to derive the expression for the Doppler effect.
C. Kikuchi (East Lansing, Mich.).

Durand, Émile. Calcul complet du rayonnement de l'oscillateur linéaire sinusoïdal. C. R. Acad. Sci. Paris 222, 68–70 (1946). [MF 15981]

The radiation from a charge in simple harmonic oscillation is considered. It is shown that the radiated frequencies are integral multiples of the period oscillation and that the amplitudes are proportional to Bessel functions of integral order. *C. Kikuchi* (East Lansing, Mich.).

King, Ronold, and Middleton, David. The cylindrical antenna; current and impedance. Quart. Appl. Math. 3, 302–335 (1946). [MF 14523]

The authors propose a new modification of Hallén's solution of the integral equation for the current in a thin cylindrical antenna. Instead of Hallén's original parameter

$\Omega=2\log 2h/a$ they introduce a variable parameter

$$\Psi(z)=\int_{-h}^{h} g(z,z')e^{-i\beta R}R^{-1}dz',$$

where $R=\{(z-z')^2+a^2\}^{\frac{1}{2}}$ and the relative distribution function $g(z,z')=I(z')/I(z)$ is to be chosen to represent the actual current distribution as closely as possible. Hallén's solution corresponds to $g(z,z')=e^{i\beta R}$. The authors use $I(z)=\sin\beta(h-|z|)$ in their definition and then take an approximate value of Ψ as a convenient parameter. Actually, their choice of parameter does not differ greatly from the average characteristic impedance $K_a=\Omega-2$ used by Schelkunoff in a different approach to the antenna problem.

First and second order terms in the expansions of the current and the input impedance are obtained in the usual way and curves are drawn for values of Ψ corresponding to $\Omega=10$, 15, 20. The results are more nearly in agreement with the available experimental evidence, especially in the neighborhood of maximum input resistance, and the second order terms are smaller than those found by Bouwkamp for Hallén's original expansion. But as yet the expansion has not been carried beyond the second order terms, so that there is no real check on the accuracy obtainable by the Hallén method. *M. C. Gray* (New York, N. Y.).

Middleton, David, and King, Ronold. The thin cylindrical antenna: a comparison of theories. J. Appl. Phys. 17, 273–284 (1946). [MF 15902]

The authors compare the various solutions of the antenna problem based on Hallén's integral equation. Hallén's own solution [Nova Acta Soc. Sci. Upsaliensis (4) 11, 1–44 (1938)] gave first order terms in the expansions of current and impedance in inverse powers of the parameter Ω and Bouwkamp [Physica **9**, 609–631 (1942); these Rev. **5**, 163] computed the second order terms in Hallén's expansion. Gray [J. Appl. Phys. **15**, 61–65 (1944); these Rev. **6**, 282] suggested a modification of Hallén's solution, introducing a variable parameter in place of Ω, and computing only first order terms. The present authors [see the preceding review] proposed a new variable parameter and used Hallén's and Bouwkamp's original computations to obtain first and second order terms. They have also modified Gray's solution to bring it into line with the others. A clear outline of the various solutions is given, and curves are included showing the input impedance obtained by each method.

Since the Hallén equation is based on a highly idealized antenna system, comparison with experimental results can only be tentative, but at present closest agreement is shown by the King-Middleton solution. The reviewer would like to suggest, however, that the comparison of the King-Middleton results with Schelkunoff's [Electromagnetic Waves, Van Nostrand, New York, 1943, p. 464], given at the end of the paper, is somewhat misleading since Schelkunoff's curve was based only on first order terms.

M. C. Gray (New York, N. Y.).

Schelkunoff, S. A. Principal and complementary waves in antennas. Proc. I. R. E. **34**, 23P–32P (1946).

This is the sequel to an earlier paper (a) by the author [Proc. I. R. E. **29**, 493–521 (1941)] and a companion to a more recent one (b) [Proc. I. R. E. **33**, 872–878 (1945); these Rev. **7**, 271]. The latter dealt primarily with a general discussion and improvement of the treatment of slim aerials by approximate solutions of the Oseen-Hallén equation in choosing as zero order term one which takes account of the wave retardation; the present paper is mainly concerned with direct approximations to the solution of Maxwell's equations, as in (a). This starts from the solution for infinite bi-conical antennas, which is elementary and akin to the transmission line equations. It can be amended to cover thin nonconical aerials, that is, with "slowly varying" cone angle, by approximation similar to the Wentzel-Kramers-Brillouin method, and, more generally, cylindrical aerials with finite radius and circular line source. These lead to variable "averaged" characteristic impedances. The associated guided waves progressing from and to a central source are called "principal waves." For finite length, the space is subdivided by a sphere S through the antenna ends into an internal (1) and external (2) part. It is pointed out that this subdivision is of immediate physical significance and thus transcends mathematical convenience. In (1), the solution is in terms of principal waves, that is, lowest order wave functions appropriate to the antenna geometry, in particular, fractional order sphericals for the bi-conical case; it is determined by the boundary condition $E_{\tan}=0$ along the wire and the amplitude prescribed at the center (source) and is devoid of radial electric components. In (2), the natural form of solution is the common expansion into spherical harmonics of integral order; radial electric components are present. The exact solution would be obtained by establishing complete continuity along S which, in the case described, is impossible lacking a sufficient set of disposable constants. These are provided by adding "complementary" waves in (1), which carry radial electrical components and which in a system with (1) only, that is, a conducting sphere S and thus with standing rather than travelling waves, would constitute higher modes. The matching of fields along S can be effected by two methods. The first is successive approximation, starting from the fundamental wave solution in (1) with sinusoidal current distribution (α), determining that in (2) via retarded potentials (β), expanding along S in terms of functions appropriate to (1) and re-establishing the boundary condition along the wire (γ), which yields a first correction of the sinusoidal distribution; then (β) and (γ) may be repeated ad infinitum. This corresponds to an expansion with orders of magnitude established by the powers of the "averaged" inverse wave impedance K^{-1}, (α) giving $O(K^{-1})$; (β), (γ) $O(K^{-2})$, etc.; (a) carried the analysis up to the latter point. The other method makes use of the orthogonality of the two sets of wave functions involved and matches the two incommensurable Fourier expansions along S so that exact field continuity is obtained for a finite number of elevation angles; this requires the solution of a system of linear algebraic equations. *H. G. Baerwald* (Cleveland, Ohio).

Watson, W. H. Wave-impedances and the effective cross-sections of antennas. Trans. Roy. Soc. Canada. Sect. III. (3) **39**, 33–51 (1945). [MF 15520]

The author first defines a wave-impedance as the dimensionless ratio $(A+B)/(A-B)$, where A and B are the amplitudes of outgoing and incoming waves, respectively. For a plane wave in the z-direction, for example, $u=Ae^{-ikz}+Be^{ikz}$. This is analogous to transmission line terminology where the above ratio is the ratio of the input impedance of a half-wave line to its characteristic impedance. Next the author discusses the scattering of a uniform plane wave incident on an antenna. The primary wave may be represented as the sum of converging and diverging spherical waves

$$P=ae^{-ikr\cos\theta}=a\sum_{n=0}^{\infty}\tfrac{1}{2}(2n+1)(O_n+I_n),$$

where O_n and I_n are products of Legendre polynomials and Hankel functions of the first and second kinds, respectively. The outgoing scattered wave is $S=\sum_{n=0}^{\infty} c_n O_n$, where the c_n's are determined by the boundary conditions. The total wave system is $P+S=\sum(A_n I_n+B_n O_n)$. Introducing the nth reflection coefficient $W_n=B_n/A_n$, we find $c_n=\frac{1}{2}(2n+1)a(W_n-1)$. Then the effective cross-section of the antenna is

$$\sigma=\sum_{n=0}^{\infty}(\pi/k^2)(2n+1)|W_n-1|^2.$$

These ideas are generalized to the case of a plane linearly polarized electromagnetic wave incident on an antenna, a set of coupling coefficients being introduced to represent the action of the antenna in converting incoming waves into outgoing waves of a different mode. The radiation from a center-fed half-wave antenna is discussed and modifications for treating high-gain antenna systems are outlined.

M. C. Gray (New York, N. Y.).

Parodi, Maurice. Conditions pour que des files de circuits récurrents, terminées par des circuits identiques, mais de structure différente de celle des circuits intermédiaires, possèdent des fréquences communes quelles que soient leurs longueurs. C. R. Acad. Sci. Paris **218**, 965–967 (1944). [MF 15246]

The author considers an electrical filter composed of an odd number of elements in cascade. The central portion consists of alternate copies of two different T-sections. The two end sections are identical, but they differ from the central sections. Using the methods of his monograph [Application des polynomes électrosphériques à l'étude des systèmes oscillants à un grand nombre de degrés de liberté, Mémor. Sci. Phys. no. 47, Gauthier-Villars, Paris, 1944; these Rev. **7**, 295], the author finds two relations between the five impedances involved, sufficient to insure that the entire network has cut-off frequencies in common with those of the central portion, regardless of the number of elements. These two relations determine the end sections when the central portion is given. *O. Frink* (State College, Pa.).

Parodi, Maurice, et Raymond, François. Propagation sur une ligne polyphasée symétrique quelconque. C. R. Acad. Sci. Paris **220**, 522–523 (1945). [MF 14052]

In a linear circuit of $n+1$ conductors (one chosen as reference for potentials),

$$\partial U/\partial x=-(\alpha)I, \quad \partial I/\partial x=-(\beta)U.$$

Here $U=(U_i)$ is the potential vector (with the n potential differences as components), and $I=(I_i)$ is the current vector (in the n branches), at the point x of the line; (α), (β) are square matrices. In a general circuit (α), (β) have elements depending on x; in the symmetric polyphase case, new coordinates will simultaneously throw (α) and (β) into diagonal form, and one gets

$$\partial U_k/\partial x=-z_k I_k, \quad \partial I_k/\partial x=-y_k U_k, \quad k=0, 1, \cdots, n-1,$$

where the y_k and z_k are functions of x. The authors refer to an application to the propagation of elastic waves.

A. L. Foster (Berkeley, Calif.).

Raymond, François. Remarques sur les équations de propagation sur une ligne quelconque. C. R. Acad. Sci. Paris **220**, 497–500 (1945). [MF 14049]

The equations of propagation along a coaxial cable are written in the form

$$(1) \qquad dV/dx=-\alpha I, \quad dI/dx=-\beta V,$$

where V, I are the Laplace transforms of $v(x,t)$, the potential at abscissa x and time t, and $i(x,t)$, the current:

$$V=\int_0^\infty e^{-pt}v\,dt, \quad I=\int_0^\infty e^{-pt}i\,dt,$$

p being linear in α, β, with coefficients depending on the capacity, impedance, etc. Vectorially (1) becomes (2) $dP=-MP\,dx$, where P is the vector (V, I) and M is the matrix

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix};$$

(2) is further written (3) $dP=-MTP_0$, with $P_0=(V_0, I_0)$ representing the state $x=0$. The author is concerned with the calculation of the coefficient (matrix) T. The complicated expression obtained for T is shown to reduce to that obtained by J. Ville for the special case where $\gamma=(\alpha\beta)^{\frac{1}{2}}$ = constant along the circuit. *A. L. Foster.*

Ingram, W. H., and Cramlet, C. M. On the foundations of electrical network theory. J. Math. Phys. Mass. Inst. Tech. **23**, 134–155 (1944). [MF 11140]

Part I of the paper, by Ingram, is devoted to the topological foundations of electrical networks. Graphs and trees are defined, a "nonseparable" graph is defined as one in which each proper sub-graph is connected to its complement at more than a single vertex, and a branch complementary to a tree of a nonseparable graph is called a "chord." It is proved that for any tree Kirchhoff's second law implies circuital flow of the chord currents; consequently "no current can flow between a separable part of a network and its complement, nor in any branch which is not spanned directly or indirectly by some other branch, nor in a network whose graph is its own tree." Connections with dynamics and Laplace transforms are discussed.

Part II of the paper is by Cramlet. Starting with a definition of a "first incidence matrix" of the graph, set up in the usual way, a "second incidence matrix" is based on circuits that are independent linear combinations of the branches. Elementary connectivity theory is then applied, employing both the integers and integers modulo 2 as base. The paper concludes with an investigation of "R-type networks," for which results on the number of topologically independent circuits are given. *H. Wallman* (Cambridge, Mass.).

Lowry, H. V. The application of the characteristic equation of a matrix to the evaluation of the range of frequencies for which currents are passed through networks with four or more terminals without attenuation. Philos. Mag. (7) **36**, 255–264 (1945). [MF 15748]

Bruges, W. E. Evaluation and application of certain ladder-type networks. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 175–186 (1946). [MF 15899]

The author derives in a novel way formulas for the overall impedance of terminated ladder networks, for finite and continuously distributed impedances, and makes applications to lengths of conductors laid in slots cut in a mass of highly permeable material, as found, for example, in rotating electrical machinery. In these applications, the formulas involve trigonometric and hyperbolic functions of the square root of a pure reactance, that is, of real multiples of $(-1)^{\frac{1}{2}}$, and hence curves of such functions, not found elsewhere, are included. *L. C. Hutchinson* (New York, N. Y.).

Pidduck, F. B. Electrical notes. XII. Alternating currents in networks. Quart. J. Math., Oxford Ser. **17**, 63–64 (1946). [MF 15897]

This is a continuation of note IV [same Quart. **2**, 174–176 (1931)].

Quantum Mechanics

Mandelstam, L., and Tamm, Ig. The uncertainty relation between energy and time in non-relativistic quantum mechanics. Acad. Sci. USSR. J. Phys. **9**, 249–254 (1945). [MF 14517]

The authors derive an uncertainty relation of the form (1) $\Delta H\cdot\Delta T\geqq h/2$ (where $2\pi h$ is Planck's constant) between energy and time. This corresponds to Heisenberg's uncertainty relation for coordinates and momenta. The present formulation of this relation is more general and more precise than some other attempts in this direction. In (1), ΔH is the standard deviation of the total energy H of an isolated system and ΔT is the shortest time during which a dynamical quantity or observable R is changed by an amount equal to its standard deviation ΔR. The relation (1) is obtained as a consequence of the more general inequality (2) $\Delta H\cdot\Delta R\geqq|\partial\bar{R}/\partial t|h/2$, where $\partial\bar{R}/\partial t$ is the rate of change of the average value $\bar{R}$ of R. This uncertainty relation is then applied to three illustrative cases: the precision of the localization in time of the passage of a wave packet through a point of space and its dependence on the dispersion of the total energy of the packet, the time of transition of energy in elastic collisions of particles and the width of spectral lines. It is shown that relations (1) and (2) hold for "mixtures" as well as for pure states described by a single wave function. *O. Frink* (State College, Pa.).

Chandrasekhar, S., and Breen, Frances Herman. The motion of an electron in the Hartree field of a hydrogen atom. Astrophys. J. **103**, 41–70 (1946). [MF 15514]

The authors consider the motion of an electron in a field whose potential energy $V(r)$ in Hartree atomic units is $V(r)=-(1+1/r)e^{-2r}$. This problem reduces to the solution of a set of second order differential equations of the form

$$\chi_l''+\{k^2-l(l+1)/r^2+2(1+1/r)e^{-2r}\}\chi_l=0$$

with the boundary conditions

$$\chi_l(0)=0,\quad \chi_l(r)\to\sin(kr-(l+1)\pi/2+\delta_l).$$

The equations are solved numerically for $l=0, 1$. The phase shift δ_0 is tabulated for $.001<k^2<1.75$; δ_1 for $.015<k^2<1.75$. The functions $\chi_0(r)$ and $\chi_1(r)$ are given for these values of k^2 and for $r=0(.1)10$. *H. Feshbach* (Cambridge, Mass.).

Eliezer, C. Jayaratnam. A discussion on the exactness of the Lorentz-Dirac classical equations. Bull. Calcutta Math. Soc. **37**, 125–130 (1945). [MF 16160]

Chang, T. S. A note on the Hamiltonian equations of motion. Proc. Cambridge Philos. Soc. **42**, 132–138 (1946). [MF 15668]

In this paper it is shown how it is possible to generalize the treatment, given earlier by F. Bopp [Ann. Physik (5) **38**, 354–384 (1940); these Rev. **2**, 336] and B. Podolsky and C. Kikuchi [Phys. Rev. (2) **65**, 228–235 (1944); these Rev. **5**, 277], of a field whose Lagrangian depends upon the second and higher derivatives of the field potentials. Field Lagrangians in which all the generalized momenta P_a are independent, not all independent, and, finally, in which some are missing, are discussed. *C. Kikuchi.*

Havas, Peter. On the interaction of radiation and matter. Phys. Rev. (2) **68**, 214–226 (1945). [MF 14113]

In a previous paper [Phys. Rev. (2) **66**, 69–76 (1944); these Rev. **6**, 224], the author derived a formula for the retarded interaction of two electrons in the presence of radiation, on the basis of the quantum theory of radiation [see W. Heitler, The Quantum Theory of Radiation, Oxford University Press, 1936]. In the present paper he derives a simpler result under more general conditions, by an extension of his previous methods based on first order approximations of perturbation theory. He obtains a simplified formula for the retarded interaction of any two particles of a collection of charged particles, due to the emission and absorption of light quanta. It has the same form as the formula of Møller, which was originally derived on different assumptions. It has the same form for all processes of emission and absorption, and is relativistically invariant. *O. Frink* (State College, Pa.).

Yamasaki, Zyunpei. On the electric field in the complex coordinates. Proc. Phys.-Math. Soc. Japan (3) **24**, 821–827 (1942). [MF 15045]

The author proposes a novel way of avoiding the infinite self-energies and other divergent expressions that arise when elementary particles are treated as point charges. He uses complex numbers for the coordinates of particles, while retaining real coordinates for points free of charge. The real parts of these complex coordinates are identical with the ordinary coordinates of the particle, while the imaginary parts are constants proportional to the mass of the particle. Complex values are also assigned to the energy and momentum of a particle. It is shown that the longitudinal and transverse self-energies of an electron are finite in this theory. It is also shown that the divergent expressions, which usually appear in second and higher order approximations of perturbation theory when applied to the quantum theory of radiation, are avoided by the author's method. *O. Frink* (State College, Pa.).

Montgomery, D. J. Relativistic interaction of electrons on Podolsky's generalized electrodynamics. Phys. Rev. (2) **69**, 117–124 (1946). [MF 15536]

The author uses the Dirac wave equation of a charged particle in a generalized field, given by B. Podolsky and C. Kikuchi [Phys. Rev. (2) **67**, 184–192 (1945); these Rev. **6**, 283], to derive the wave equation of a system of particles. The method is based on a scheme suggested by V. Fock [Phys. Z. Sowjetunion **6**, 425–469 (1934)]. The wave equation is then used to obtain a generalization of Møller's formula [Z. Physik **70**, 786–795 (1931)]. *C. Kikuchi.*

Becker, R., and Leibfried, G. On the method of second quantization. Phys. Rev. (2) **69**, 34 (1946). [MF 15214]

In this very brief note the quantities $\psi(x)$ are treated as operators instead of eigenfunctions. The elements in Hilbert space on which they operate are exhibited. Commutation rules for these operators are given, both for Bose-Einstein and Fermi-Dirac statistics. *O. Frink.*

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